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Nash Equilibria in Quantum Games

by

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Quantum game theory investigates the behavior of strategic agents with access to quantum technology, such as the quantum computers that might or might not be standard office equipment in the fairly near future. Just as players equipped with ordinary dice can implement mixed strategies, so players equipped with quantum devices can implement "quantum strategies" that referees or regulators would find difficult to prohibit.

The optimal use of quantum technology depends on the details of the interaction between players and their environment. It therefore becomes important to model the means by which the players communicate their strategies (either to each other, or to their customers, or their suppliers, or some other third party). In the most widely studied model, Eisert, Wilkens and Lewenstein ([EW], [EWL]) posit a referee to serve as a metaphor for the recipients of that communication. The communication itself takes place according to a pre-arranged protocol and players manipulate their messages to take advantage of that protocol.

In the (two by two) EWL model, each player starts with a "penny" (actually a subatomic particle) which he returns unflipped to indicate a play of strategy **C** or flipped to indicate a play of strategy **D**. Classically, a single penny must be either in the state **H** (heads up) or **T**, (tails up), and a pair of pennies is in one of four states $\mathbf{H}\otimes\mathbf{H}$, $\mathbf{H}\otimes\mathbf{T}$, etc. Quantum mechanics allows many additional possibilities. In the EWL model, the pennies start out in the *maximally entangled state* $\mathbf{H}\otimes\mathbf{H}+\mathbf{T}\otimes\mathbf{T}$. When inspected for orientation, such pennies are found either (heads,heads) or (tails,tails), each with probability 1/2. However, the referee doesn't make that observation; instead of observing the heads/tails variable, he observes flipped/unflipped. (For classical pennies, these are equivalent observations; for quantum pennies they are emphatically not.) This is of key importance, because it means players can still play their classical strategies **C** and **D**; the original game is imbedded in the quantum extension.

Where do the entangled pennies come from? Depending on the story one wants to tell, they might be issued by the referee or devised by the players in pre-game cheap talk. (The physical manufacture of entangled particles is an everyday occurrence in physics labs.) Each player can always discard his entangled penny and substitute an unentangled quantum penny (or for that matter an ordinary classical penny) but in equilibrium they won't want to (see Remark 4.2).

The EWL model is of particular interest because it captures, in a fairly general setting, the behavior of players who use quantum technology to manipulate their communications, taking as given the protocol by which those communications will be deciphered.

In the EWL model, the strategy spaces naturally expand from the two-point space $\{\mathbf{C},\mathbf{D}\}$ to the space of unit quaternions. A quaternion is an expression of the form $A+Bi+Cj+Dk$ with $A,B,C,D\in\mathbf{R}$ and i,j,k symbols satisfying $i^2=j^2=k^2=ijk=-1$. Thus quaternions are naturally identified with points in $\mathbf{R}^4$ and the *unit quaternions* are defined to be those that lie on the three- dimensional unit sphere $\mathbf{S}^3$. The space of *mixed quantum strategies* is therefore the space of arbitrary probability distributions on $\mathbf{S}^3$. In principle, the vastness of this space makes it difficult to compute equilibria or to know when you've found them all. Quite a bit of attention has been lavished on identifying equilibria in particular two by two EWL games.

The chief contribution of this paper is Theorem 3.3, which classifies all Nash equilibria up to a natural notion of equivalence, and to show that they are exceptionally simple. In equilibrium, each player's mixed strategy is supported on at most four points. Moreover, these points must lie in certain quite restrictive geometric configurations. This transforms the search for equilibria from a potentially intractable problem into an almost mechanical one.

In Section 1, I will lay out the EWL model. In Section 2, I will present the main technical lemmas and in Section 3 I will prove the main theorems. Section 4 contains some easy applications; the most striking is that in any mixed strategy quantum equilibrium of any two-by-two zero sum game, each player earns exactly the average of the four possible payoffs.

1. The Eisert-Wilkens-Lewenstein Model.

A classical penny can be either in the state $\mathbf{H}$ ("heads") or the state $\mathbf{T}$ ("tails"). The states of a quantum penny are represented by expressions $\alpha\mathbf{H}+\beta\mathbf{T}$ where α and β are complex numbers, not both zero. Two such expressions represent the same state if (and only if) one is a (complex) scalar multiple of the other.

An *entangled pair* of pennies is in a state represented by a non-zero expression

$$\alpha\mathbf{H}\otimes\mathbf{H}+\beta\mathbf{H}\otimes\mathbf{T}+\gamma\mathbf{T}\otimes\mathbf{H}+\delta\mathbf{T}\otimes\mathbf{T}$$

where the coefficients are complex numbers, and where a scalar multiple represents the same state.

Start with a two-by-two game $\mathbf{G}$ where each player's strategy space is $\{\mathbf{C},\mathbf{D}\}$. Each player acquires (or is issued) one of two entangled pennies, which start out in the *maximally entangled state*

$$\mathbf{H}\otimes\mathbf{H}+\mathbf{T}\otimes\mathbf{T}$$

They return their pennies unflipped to indicate a play of $\mathbf{C}$ or flipped to indicate a play of $\mathbf{D}$. After the pennies are returned, the referee observes whether they've been flipped and makes payoffs accordingly.

As long as players obediently play either $\mathbf{C}$ or $\mathbf{D}$, the game remains unchanged. But quantum mechanics allows players to act on their pennies by arbitrary *special unitary matrices*, that is, complex two by two matrices of determinant one satisfying $M^{-1}=\overline{M}^T$ (where the overbar indicates complex conjugation and the superscript T indicates the transpose). Acting by the identity matrix means leaving the penny unflipped. Acting by the matrix

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\tag{1.1}$$

means flipping the penny. Other unitary matrices correspond to physical operations with no classical analogues.

If Players One and Two act by the unitary matrices

$$\begin{pmatrix}A&B\\-\overline{B}&\overline{A}\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}P&Q\\-\overline{Q}&\overline{P}\end{pmatrix}$$

then the pennies come back to the referee in the state

$$(A\mathbf{H}-\overline{B}\mathbf{T})\otimes(P\mathbf{H}-\overline{Q}\mathbf{T})+(B\mathbf{H}+\overline{A}\mathbf{T})\otimes(Q\mathbf{H}+\overline{P}\mathbf{T})$$

which expands to

$$(AP+BQ)\mathbf{H}\otimes\mathbf{H}+(-A\overline{Q}+B\overline{P})\mathbf{H}\otimes\mathbf{T}+(-\overline{B}P+\overline{A}Q)\mathbf{T}\otimes\mathbf{H}+(\overline{AP}+\overline{BQ})\mathbf{T}\otimes\mathbf{T}\tag{1.2}$$

If the players choose classical strategies (either the identity matrix or the matrix (1.1)), the pennies come back to the referee in one of four states

$$\mathbf{CC} = \mathbf{H} \otimes \mathbf{H} + \mathbf{T} \otimes \mathbf{T} \tag{1.3a}$$

$$\mathbf{CD} = \mathbf{H} \otimes \mathbf{T} + \mathbf{T} \otimes \mathbf{H} \tag{1.3b}$$

$$\mathbf{DC} = \mathbf{H} \otimes \mathbf{T} - \mathbf{T} \otimes \mathbf{H} \tag{1.3c}$$

$$\mathbf{DD} = \mathbf{H} \otimes \mathbf{H} - \mathbf{T} \otimes \mathbf{T} \tag{1.3d}$$

and the referee makes payoffs accordingly. If players adopt more general strategies, returning the pennies in state (1.2), the referee's observation causes them to collapse into one of the states (1.3a-d) with probabilities calculated (according to the laws of quantum mechanics) as follows:

First write (1.2) as a linear combination

$$\alpha\mathbf{CC} + \beta\mathbf{DD} + \gamma\mathbf{CD} + \delta\mathbf{DC} \tag{1.4}$$

(with complex scalar coefficients). Then the probabilities of the four states are proportional to $|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2$.

Notice that the referee cannot detect (and therefore cannot prohibit) the play of nonclassical strategies. All the referee ever observes is a pair of pennies in one of the states (1.3a-d).

From (1.2) one readily calculates the coefficients in (1.4) and discovers (remarkably) that they are real numbers; for example, α is twice the real part, and δ twice the imaginary part, of $AP + BQ$.

It's then easy to check the following:

Proposition 1.1. *When expression (1.2) is written in the form (1.4), the coefficients are proportional to the components of the unit quaternion* $(A + Bj)(P - jQ)$.

Motivated by Proposition 1.1 and the preceding discussion, we make the following definitions:

Definitions 1.2. Let $\mathbf{G}$ be a two by two game with strategy spaces $S_i = \{\mathbf{C}, \mathbf{D}\}$ and payoff functions $P_i : S_1 \times S_2 \to \mathbf{R}$. Then the *associated quantum game* $\mathbf{G}^Q$ is the two-player game in which each strategy space is the unit quaternions, and payoffs are calculated as

$$P_i^Q(\mathbf{p}, \mathbf{q}) = \pi_1(\mathbf{pq})^2 P_i(\mathbf{C}, \mathbf{C}) + \pi_2(\mathbf{pq})^2 P_i(\mathbf{D}, \mathbf{D}) + \pi_3(\mathbf{pq})^2 P_i(\mathbf{C}, \mathbf{D}) + \pi_4(\mathbf{pq})^2 P_i(\mathbf{D}, \mathbf{C})$$

where the π_t are the coordinate functions defined by $\mathbf{p} = \pi_1(\mathbf{p}) + \pi_2(\mathbf{p})i + \pi_3(\mathbf{p})j + \pi_4(\mathbf{p})k$.

A *mixed quantum strategy* for $\mathbf{G}$ is a mixed strategy in the game $\mathbf{G}^Q$, i.e. a probability distribution on the space of unit quaternions.

If $\mathbf{p}$ is a unit quaternion, I will sometimes identify $\mathbf{p}$ with the mixed strategy supported entirely on $\mathbf{p}$. If ν and μ are mixed strategies, I will write $P_i^Q(\nu, \mu)$ for the corresponding expected playoff to player i; that is:

$$P_i^Q(\nu, \mu) = \int P_i(\mathbf{p}, \mathbf{q}) d\nu(\mathbf{p}) d\mu(\mathbf{q})$$

Our goal is to classify the Nash equilibria in $\mathbf{G}^Q$. The definitions that occupy the remainder of this section kick off that process by partitioning the set of Nash equilibria into natural equivalence classes.

Definition 1.3. Two mixed strategies μ and μ' are *equivalent* if

$$\int \pi_t(\mathbf{pq}) d\mu(\mathbf{q}) = \int \pi_t(\mathbf{pq}) d\mu'(\mathbf{q})$$

for all unit quaternions $\mathbf{p}$ and all $t = 1, 2, 3, 4$.

In other words, μ and μ' are equivalent if in every quantum game and for every quantum strategy $\mathbf{p}$, we have $P_1(\mathbf{p}, \mu) = P_1(\mathbf{p}, \mu')$ and $P_2(\mathbf{p}, \mu) = P_2(\mathbf{p}, \mu')$.

Example 1.4. The strategy supported on the singleton $\{\mathbf{p}\}$ is equivalent to the strategy supported on the singleton $\{-\mathbf{p}\}$ and to no other singleton.

Definition 1.5. Let ν be a mixed strategy and $\mathbf{u}$ a unit quaternion. The *right translate* of ν by $\mathbf{u}$ is the measure $\nu\mathbf{u}$ definied by $(\nu\mathbf{u})(A) = \nu(A\mathbf{u})$ where A is any subset of the unit quaternions and $A\mathbf{u} = \{\mathbf{xu} | x \in A\}$. Similarly, the left translate of ν by $\mathbf{u}$ is defined by $(\mathbf{u}\nu)(A) = \nu(\mathbf{u}A)$. The following proposition is immediate:

Proposition 1.6. Let (ν, μ) be a pair of mixed strategies and $\mathbf{u}$ a unit quaternion. Then in any game $\mathbf{G}^Q$, (ν, μ) is a mixed strategy Nash equilibrium if and only if $(\nu\mathbf{u}, \mathbf{u}^{-1}\mu)$ is.

Definition 1.7. Two pairs of mixed strategies (ν, μ) and (ν', μ') are *equivalent* if there exists a unit quaternion $\mathbf{u}$ such that ν' is equivalent to $\nu\mathbf{u}$ and μ' is equivalent to $\mathbf{u}^{-1}\mu$. Note that this definition is independent of any particular game.

Proposition 1.8. In a given game, a pair of mixed strategies is a Nash equilibrium if and only if every equivalent pair of mixed strategies is also a Nash equilibrium.

2. Preliminary Results.

Theorems 2.1, 2.2 and 2.4 are the main results which will be used in Section 3 to classify Nash equilibria.

Theorem 2.1. Every mixed strategy is equivalent to a mixed strategy supported on (at most) four points. Those four points can be taken to form an orthonormal basis for $\mathbf{R}^4$.

Proof. First, choose any orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ for $\mathbf{R}^4$. For any quaternion $\mathbf{p}$, write (uniquely)

$$\mathbf{p} = \sum_{\alpha=1}^{4} A_\alpha(p)\mathbf{q}_\alpha$$

where the $A_\alpha(p)$ are real numbers.

Define a probability measure ν supported on the four points $\mathbf{q}_\alpha$ by

$$\nu(\mathbf{q}_\alpha) = \int_{\mathbf{S}^3} A_\alpha(\mathbf{q})^2 d\mu(\mathbf{q})$$

For any two quaternions $\mathbf{p}$ and $\mathbf{q}$, define

$$X(\mathbf{p}, \mathbf{q}) = \sum_{\alpha=1}^{4} \pi_\alpha(\mathbf{p})\pi_\alpha(\mathbf{q})X_i \tag{2.1.1}$$

Then for any $\mathbf{p}$ we have

$$\begin{aligned}
P(\mathbf{p}, \mu) &= \int_{\mathbf{S}^3} P(\mathbf{pq}) d\mu(\mathbf{q}) \\
&= \int_{S^3} P\left(\sum_{\alpha=1}^{4} A_\alpha(\mathbf{q})\mathbf{p}\mathbf{q}_\alpha\right) d\mu(\mathbf{q}) \\
&= \sum_{\alpha=1}^{4} P(\mathbf{p}\mathbf{q}_\alpha) \int_{S^3} A_\alpha(\mathbf{q})^2 d\mu(\mathbf{q}) + 2\sum_{\alpha\neq\beta} X(\mathbf{p}\mathbf{q}_\alpha, \mathbf{p}\mathbf{q}_\beta) \int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q}) d\mu(\mathbf{q}) \\
&= P(\mathbf{p}, \nu) + 2\sum_{\alpha\neq\beta} X(\mathbf{p}\mathbf{q}_\alpha, \mathbf{p}\mathbf{q}_\beta) \int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q}) d\mu(\mathbf{q})
\end{aligned}$$

To conclude that μ is equivalent to ν it is sufficient (and necessary) to choose the $\mathbf{q}_\alpha$ so that for each $\alpha \neq \beta$ we have

$$\int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q}) d\mu(\mathbf{q}) = 0$$

For this, consider the function $B : \mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}$ defined by

$$B(\mathbf{a}, \mathbf{b}) = \int_{\mathbf{S}^3} \pi_1(\overline{\mathbf{a}}\mathbf{q})\pi_1(\overline{\mathbf{b}}\mathbf{q}) d\mu(\mathbf{q})$$

B is a bilinear symmetric form and so can be diagonalized; take the $\mathbf{q}_\alpha$ to be an orthonormal basis with respect to which B is diagonal. Then we have (for $\alpha \neq \beta$)

$$\begin{aligned}\int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q})d\mu(\mathbf{q}) &= \int_{\mathbf{S}^3} \pi_1(\overline{\mathbf{q}_\alpha}\mathbf{q})\pi_1(\overline{\mathbf{q}_\beta}\mathbf{q})d\mu(\mathbf{q}) \\ &= B(\mathbf{q}_\alpha, \mathbf{q}_\beta) = 0\end{aligned}$$

■

Theorem 2.2. Taking Player 2's (mixed) strategy μ as given, Player 1's optimal response set is equal to the intersection of $\mathbf{S}^3$ with a linear subspace of $\mathbf{R}^4$.

(Recall that we identify the unit quaternions with the three-sphere $\mathbf{S}^3$.)

Proof. Player One's problem is to choose $\mathbf{p} \in S^3$ to maximize

$$P_1(\mathbf{p}, \mu) = \int P_1(\mathbf{pq})d\mu(\mathbf{q}) \tag{2.2.1}$$

Expression (2.2.1) is a (real) quadratic form in the coefficients $\pi_i(\mathbf{p})$ and hence is maximized (over S^3) on the intersection of S^3 with the real linear subspace of $\mathbf{R}^4$ corresponding to the maximum eigenvalue of that form.

■

Definition 2.3. We define the function $K : \mathbf{S}^3 \to \mathbf{R}$ by $K(A + Bi + Cj + Dk) = ABCD$. Thus in particular $K(\mathbf{p}) = 0$ if and only if $\mathbf{p}$ is a linear combination of at most three of the fundamental units $\{1, i, j, k\}$.

Theorem 2.4. Let μ be a mixed strategy supported on four orthogonal points $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ played with probabilities $\alpha, \beta, \gamma, \delta$. Suppose $\mathbf{p}$ is an optimal response to μ in some game where it is not the case that $X_1 = X_2 = X_3 = X_4$. Then $\mathbf{p}$ must satisfy:

$$\begin{aligned}&(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)K(\mathbf{pq}_1) + (\beta-\alpha)(\beta-\delta)(\beta-\gamma)K(\mathbf{pq}_2) \\ &+(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)K(\mathbf{pq}_3) + (\delta-\alpha)(\delta-\beta)(\delta-\gamma)K(\mathbf{pq}_4) = 0\end{aligned} \tag{2.4.1}$$

Proof. Set $\mathbf{p}_n = \pi_n(\mathbf{p})$ and consider the function

$$\begin{array}{rccc}\mathcal{P}: & \mathbf{S}^3 \times \mathbf{R}^4 & \to & \mathbf{R} \\ & (\mathbf{p}, \mathbf{x}) & \mapsto & \sum_{n=1}^4 \mathbf{p}_n^2 \mathbf{x}_n d\mu(\mathbf{q})\end{array}$$

In particular, if we let $X = (X_1, X_2, X_3, X_4)$ then $\mathcal{P}(\mathbf{p}, X) = P_1(\mathbf{p}, \mu)$.

The function $\mathcal{P}$ is quadratic in $\mathbf{p}$ and linear in $\mathbf{x}$; explicitly we can write

$$\mathcal{P}(\mathbf{p},\mathbf{x}) = \sum_{i,j,k} t_{ijk}\mathbf{p}_i\mathbf{p}_j\mathbf{x}_k$$

for some real numbers t_{ijk}.

Set

$$M_{ij}(\mathbf{x}) = \sum_{k=1}^{4} t_{ijk}\mathbf{x}_k$$
$$N_{ij}(\mathbf{p}) = \sum_{k=1}^{4} t_{ikj}\mathbf{p}_j$$

so that

$$M(\mathbf{x}) \cdot \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = N(\mathbf{p}) \cdot \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \tag{2.4.2}$$

If $\mathbf{p}$ is an optimal response to the strategy μ, then $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)^T$ must be an eigenvector of $M(X)$, say with associated eigenvalue λ. From this and (2.4.2) we conclude that

$$N(\mathbf{p}) \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \lambda \cdot \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = N(\mathbf{p}) \cdot \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \lambda \end{pmatrix}$$

where the second equality holds by an easy calculation.

Thus $N(\mathbf{p})$ must be singular. But it follows from a somewhat less easy calculation that the determinant of $N(\mathbf{p})/2$ is given by the left side of (2.4.1).

3. Classification.

Definition 3.1. Let $\mathbf{G}$ be a two-by-two game with payoff pairs $(X_1,Y_1),\ldots,(X_4,Y_4)$ (listed in arbitrary order). $\mathbf{G}$ is a *generic game* if the X_i are all distinct, the Y_i are all distinct, the twofold sums $X_i + X_j$ are all distinct and the twofold sums $Y_i + Y_j$ are all distinct.

Theorem 3.3 will classify Nash Equilibria in $\mathbf{G}^Q$ where $\mathbf{G}$ is any generic two-by-two game. Subtler versions of the same arguments work for non-generic games; see [NE].

To state Theorem 3.3 we need a definition:

Definition 3.2. Let $\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}$ be quaternions; write $\mathbf{p} = p_1 + p_2 i + p_3 j + p_4 k$, etc. Then the quadruple $(\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s})$ is *intertwined* if there is a nonzero constant α such that

$$\alpha(X\mathbf{p} + Y\mathbf{q}) = X\mathbf{r} + Y\mathbf{s}$$

identically in X and Y.

Thus if the components of $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ are all nonzero, then $(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})$ is intertwined if and only if the four quotients $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}$ are equal (in some order) to the four quotients $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3}, \frac{r_4}{s_4}$.

The intertwined quadruple $(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})$ is *fully intertwined* if $(\mathbf{p}, \mathbf{r}, \mathbf{q}, \mathbf{s})$ is also intertwined.

Fully intertwined quadruples are classified in [I], where the bottom line is that almost all fully intertwined quadruples have one of a few easily recognizable forms. The exceptions are very rare, not just in the sense of having measure zero but in the (stronger) sense of having high codimension in the space of fully intertwined quadruples.

We can now state the main theorem:

Theorem 3.3. Let $\mathbf{G}$ be a generic game. Then up to equivalence, every equilibrium in $\mathbf{G}^Q$ is of one of the following types:

a) Each player plays each of four orthogonal quaternions with probability $1/4$.

b) Each player's strategy is supported on three of the four quaternions $1, i, j, k$.

c) μ is supported on two orthogonal points $1, \mathbf{v}$; ν is supported on two orthogonal points $\mathbf{p}, \mathbf{pu}$, and the quadruple $(\mathbf{p}, \mathbf{pv}, \mathbf{pu}, \mathbf{pvu})$ is fully intertwined.

d) Each of μ and ν is supported on two orthogonal points, each played with probability $1/2$. Moreover, the supports of μ and ν lie in parallel planes.

e) Each player plays a pure strategy from the four point set $\{1, i, j, k\}$.

Proof. Let (ν, μ) be an equilibrium. By (2.1) we can assume that each of ν and μ is supported on a set of at most four orthogonal points. Applying a translation as in (1.7) we can assume that the support of μ contains the quaternion 1. Then from standard facts about orthogonality in the space of quaternions, the support of μ is contained in a set of the form $\{1, \mathbf{u}, \mathbf{v}, \mathbf{uv}\}$ where $\mathbf{u}^2 = \mathbf{v}^2 = -1$ and $\mathbf{uv} + \mathbf{vu} = 0$, played with probabilities of $\alpha, \beta, \gamma, \delta \geq 0$. We will maintain these assumptions and this notation while proving Theorems 3.4, 3.5, 3.9, and 3.10, which together imply Theorem 3.3.

Theorem 3.4. ν is a pure strategy if and only if μ is a pure strategy.

Proof. If ν is a pure strategy, Player Two can guarantee any desired probability distribution over four outcomes; by genericity his optimal probability distribution is unique. ∎

Theorem 3.5. If the support of ν contains four points then μ assigns probability 1/4 to each of four strategies.

Corollary 3.5A. If either player's strategy has a four-point support, then each player plays each of four orthogonal quaternions with probability 1/4.

Proof of Corollary. Apply Theorem 3.5 twice, one as stated and once with the players reversed.

Proof of Theorem 3.5. Explicitly write $\mathbf{u} = Ai + Bj + Ck, \mathbf{v} = Di + Ej + Fk, \mathbf{uv} = Gi + Hj + Ik.$ Write

$$\mathcal{M} = \begin{pmatrix} AB & DE & GH \\ AD & DF & GI \\ BC & EF & HI \end{pmatrix}$$

By (2.2) the quadratic form

$$\mathbf{p} \mapsto P_1(\mathbf{p}, \mu) \tag{3.5.1}$$

is constant on the unit sphere $\mathbf{S}^3$. Therefore its non-diagonal coefficients are all zero. Computing these coefficients explicitly and dividing by (non-zero) expressions of the form $(x_i - x_j)$, we get

$$\mathcal{M} \cdot (\beta, \gamma, \delta)^T = (0, 0, 0)^T \tag{3.5.2}$$

But $\mathcal{M}$ also kills the column vector $(1, 1, 1)^T$. Thus we have two cases:

Case I. $\beta = \gamma = \delta$. Then the four diagonal terms of (3.5.1) (which must all be equal) are given by $(X_1 + X_2 + X_3 + X_4)\beta + X_i(\alpha - \beta)$, with $i = 1, 2, 3, 4$. Since the X_i are not all equal, it follows $\alpha = \beta = \gamma = \delta = 1/4$, proving the theorem.

Case II. M has rank at most one. From this and the orthogonality of **u**,**v**,**uv**, we have $\{\mathbf{u}, \mathbf{v}, \mathbf{uv}\} \cap \{i, j, k\} \neq \emptyset$. Assume $\mathbf{u} = i$ (the other cases are similar). Then $A = 1$, $B = C = D = G = 0$, $H = -F$ and $I = E$. The four diagonal entries of (3.3.1) are now equal; call their common value λ so that we have

$$\begin{pmatrix} \alpha & \beta & E^2\gamma + F^2\delta & E^2\delta + F^2\gamma \\ \beta & \alpha & E^2\delta + F^2\gamma & E^2\gamma + F^2\delta \\ E^2\gamma + F^2\delta & E^2\delta + F^2\gamma & \alpha & \beta \\ E^2\delta + F^2\gamma & E^2\gamma + F^2\delta & \beta & \alpha \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \lambda \end{pmatrix} \tag{3.5.3}$$

Combining (3.5.2), (3.5.3), the conditions $\alpha + \beta + \gamma + \delta = E^2 + F^2 = 1$ and the genericity conditions, we get $\alpha = \beta = \gamma = \delta$ as required. ∎

Theorem 3.9, dealing with the case where ν is supported on exactly three points, requires some preliminary lemmas:

Lemma 3.6. It is not the case that Player Two plays $1, \mathbf{u}, \mathbf{v}$ each with probability $1/3$.

Proof. If $1, \mathbf{u}, \mathbf{v}$ are played with probability $1/3$ then one computes that the eigenvalues of the form (2.2.1) are $X_1 + X_2 + X_3$, $X_1 + X_2 + X_4$, $X_1 + X_3 + X_4$, $X_2 + X_3 + X_4$, which are all distinct by genericity. Thus Player One responds with a pure strategy, and Theorem 3.4 provides a contradiction. ∎

Lemma 3.7. Suppose the support of ν is contained in the linear span of $1, i, j$. and suppose that 1 and i are both optimal responses for Player Two. Then one of the following is true:

a) The support of ν is contained in the three point set $\{1, i, j\}$

b) The support of ν is contained in a set of the form $\{1, Ei+Fj, -Fi+Ej\}$ with $Ei+Fj$ and $-Fi+Ej$ played equiprobably.

Moreover, if b) holds and either j or k is also an optimal response for Player Two, then 1 is played with probability zero.

Proof. Suppose ν is supported on three orthogonal quaternions $\mathbf{q}_1 = A + Bi + Cj$, $\mathbf{q}_2 = D + Ei + Fj$, $\mathbf{q}_3 = G + Hi + Ij$, played with probabilities ϕ, ψ, ξ. The first order conditions for Player Two's maximization problem must be satisfied at both 1 and i; this (together with genericity for the game $\mathbf{G}$) gives

$$\begin{pmatrix} AC & DF & GI \\ BC & EF & HI \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} AC & DF & GI \\ BC & EF & HI \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.7.1}$$

so that by (3.6) with the players reversed, the matrix on the left has rank at most one. This (together with the orthogonality of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$) gives $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} \cap \{1, i, j\} \neq \emptyset$. We can assume $\mathbf{q}_1 = 1$ (all other cases are similar); thus $A = 1$, $B = C = D = G = 0$, $H = -F$, $I = E$. Now (3.7.1) says $EF(\psi - \xi) = 0$. If $EF = 0$, then a) holds and if $\psi - \xi = 0$ then b) holds.

Now suppose j is also an optimal response for Player Two. Then $0 = P_2(\nu, i) - P_2(\nu, j) = \phi(Y_2 - Y_3)$, so that by genericity $\phi = 0$. A similar argument works if k is optimal. ∎

Lemma 3.8. Suppose ν is supported on exactly three points and continue to assume that μ is supported on a subset of $\{1, \mathbf{u}, \mathbf{v}\,\mathbf{uv}\}$. Then at least two of the four quaternions $1, \mathbf{u}, \mathbf{v}, \mathbf{uv}$ are optimal responses for Player One.

Proof. By (3.5), μ is supported on at most three points; we can rename so those points are $1, \mathbf{u}, \mathbf{v}$. These are played with probabilities α, β, γ and we can rename again so that α lies (perhaps not strictly) between β and γ.

If $\mathbf{p}$ is any optimal response by Player One, apply (2.4) with $\delta = 0$ (and possibly $\gamma = 0$) to get

$$\sigma_1 K(\mathbf{p}) + \sigma_2 K(\mathbf{pu}) + \sigma_3 K(\mathbf{pv}) + \sigma_4 K(\mathbf{puv}) = 0 \tag{3.8.1}$$

where $\sigma_1 = (\alpha - \beta)(\alpha - \gamma)\alpha$, etc., so that

$$\sigma_1, \sigma_4 \leq 0 \qquad \text{and} \qquad \sigma_2, \sigma_3 \geq 0 \tag{3.8.2}$$

Case I: Suppose none of the σ_i is equal to zero. Then $\gamma \neq 0$ so a) holds.

By (2.2), the support of ν spans a three-dimensional hyperplane in $\mathbf{R}^4$ and thus must include some quaternion of the form $A + B\mathbf{u}$ $(A, B \in \mathbf{R})$. Inserting $\mathbf{p} = A + B\mathbf{u}$ into (3.8.1) gives

$$AB(\sigma_1 B^2 - \sigma_2 A^2)K(1 + \mathbf{u}) = 0 \tag{3.8.3}$$

Thus either $AB = 0$ (in which case either $\mathbf{p} = 1$ or $\mathbf{p} = \mathbf{u}$) or $K(1 + \mathbf{u}) = 0$. This and similar arguments establish the following:

$$\text{If } 1 \text{ and } \mathbf{u} \text{ are both suboptimal responses, then } K(1 + \mathbf{u}) = 0. \tag{3.8.3a}$$

$$\text{If } 1 \text{ and } \mathbf{v} \text{ are both suboptimal responses, then } K(1 + \mathbf{v}) = 0. \tag{3.8.3b}$$

$$\text{If } \mathbf{u} \text{ and } \mathbf{uv} \text{ are both suboptimal responses, then } K(1 + \mathbf{v}) = 0. \tag{3.8.3c}$$

$$\text{If } \mathbf{v} \text{ and } \mathbf{uv} \text{ are both suboptimal responses, then } K(1 + \mathbf{u}) = 0. \tag{3.8.3d}$$

Taken together, these imply that if the lemma fails, then $K(1 + \mathbf{u}) = K(1 + \mathbf{v}) = 0$. From this it follows that $\{\mathbf{u}, \mathbf{v}, \mathbf{uv}\} \cap \{\pm i, \pm j, \pm k\} \neq \emptyset$; assume without loss of generality that $\mathbf{u} = i$ and therefore $\mathbf{v}$ is in the linear span of $\{j, k\}$. (Generality is not lost because the argument to follow works just as well, with obvious modifications, in all the remaining cases.)

Now we have

$$\begin{aligned} P_1(A + B\mathbf{v}, \mu) &= \alpha P_1(A + B\mathbf{v}) + \beta P_1(Ai + B\mathbf{v}i) + \gamma P_1(A\mathbf{v} - B) \\ &= A^2\Big(\alpha P_1(1) + \beta P_1(i) + \gamma P_1(\mathbf{v})\Big) + B^2\Big(\alpha P_1(\mathbf{v}) + \beta P_1(\mathbf{v}i) + \gamma P_1(1)\Big) \end{aligned}$$

which is maximized at an endpoint, so either 1 or $\mathbf{v}$ is an optimal response for Player One. Similarly, at least one from each pair $\{1, \mathbf{uv}\}$, $\{\mathbf{u}, \mathbf{v}\}$, and $\{\mathbf{u}, \mathbf{uv}\}$ is an optimal response, from which b) (and therefore the lemma) follows.

Case II: Suppose at least one of the σ_i is equal to zero. Up to renaming $\mathbf{u}$ and $\mathbf{v}$, there are three ways this can happen:

Subcase IIA: $\alpha = \beta$, $\gamma = 0$. As above, Player One's optimal response set contains a quaternion of the form $(A + B\mathbf{u})$. But $P_1(A + B\mathbf{u}, \mu)$ is independent of A and B, so both 1 and $\mathbf{u}$ are optimal, proving the theorem. (Note that $\mathbf{v}$ and $\mathbf{uv}$ are also both optimal, so that in fact by (3.5A) this case never occurs.)

Subcase IIB: $\alpha = \beta$, $\gamma \neq 0$. By Lemma (3.6), $\gamma \neq \alpha, \beta$. Thus σ_3 and σ_4 are nonzero, so (3.8.3b), (3.8.3c) and (3.8.3d) (but not (3.8.3a)) still hold. But $\sigma_1 = \sigma_2 = 0$ so the same techniques now yield

$$\text{If 1 and } \mathbf{u} \text{ are both suboptimal responses, then } K(1 + \mathbf{u}) = 0. \tag{3.8.3e}$$

$$\text{If 1 and } \mathbf{v} \text{ are both suboptimal responses, then } K(1 + \mathbf{v}) = 0. \tag{3.8.3f}$$

We can now repeat the argument from Case I.

Subcase IIC: $\alpha \neq \beta$, $\gamma = 0$. Now we have $\sigma_1, \sigma_2 \neq 0$, $\sigma_3 = \sigma_4 = 0$, so that (3.8.3a) through (3.8.3c) still hold, along with (3.8.3e) and (3.8.3f). We can now repeat the argument from Case I. ∎

Theorem 3.9. If ν is supported on exactly three points, then up to equivalence, both μ and ν are supported on three-point subsets of $\{1, i, j, k\}$.

Proof. By (3.5) we can assume that μ is supported on $\{1, \mathbf{u}, \mathbf{v}\}$. By (3.8) we can assume without much loss of generality that 1 and $\mathbf{u}$ are optimal responses for Player One. (The argument below works equally well, with obvious modifications, for other pairs.) Let $\mathbf{w}$ be a quaternion orthogonal to 1 and $\mathbf{u}$ such that the support of ν is contained in the linear span of $1, \mathbf{u}$ and $\mathbf{w}$.

By (2.2), any quaternion of the form $X + Y\mathbf{u} + Z\mathbf{w}$ is an optimal response for Player One, so by (2.4) we have

$$\sigma_1 K(X + Y\mathbf{u} + Z\mathbf{w}) + \sigma_2 K(X\mathbf{u} - Y + Z\mathbf{wu}) + \sigma_3 K(X\mathbf{v} + Y\mathbf{uv} + Z\mathbf{wv}) + \sigma_4 K(X\mathbf{uv} - Y\mathbf{v} + Z\mathbf{wuv}) = 0$$

identically in X, Y, Z. Writing out the left side as a polynomial in these three variables, the coefficients, all of which must vanish, can be expressed in terms of the components of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Setting all these expressions equal to zero and solving, we find that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \in \{\pm i, \pm j, \pm k\}$. (The details of this tedious but straightforward calculation can be found on pages 32-33 of [NE].) We assume $\mathbf{u} = i$, $\mathbf{w} = j$.

Claim: Player Two's strategy is not supported just on 1 and i. Proof: If so, the fact that $P_1(1,\mu) = P_1(i,\mu)$ implies that μ assigns equal weights to 1 and i, which implies $P_1(j,\mu) = P_1(k,\mu)$, contradicting the fact that j but not k is optimal for Player One.

Thus the support of μ is a three-point subset of $\{1, i, j, k\}$. It now follows from Lemma (3.8) (together with the assumption that the support of ν contains three points) that the support of ν is $\{1, i, j\}$, completing the proof. ∎

Theorem 3.10. Suppose ν is supported on two points. Then μ is supported on $1, \mathbf{u}$ and ν is supported on two quaternions $\mathbf{p}, \mathbf{pv}$ where either

a) The quadruple $(\mathbf{p}, \mathbf{pu}, \mathbf{pv}, \mathbf{pvu})$ is fully intertwined or

b) $\mathbf{u} = \mathbf{v}$ and each player plays each strategy with probability $1/2$.

Proof. Suppose 1 and $\mathbf{u}$ are played with probabilities α and β.

Any unit quaternion of the form $X\mathbf{p} + Y\mathbf{pv}$ is an optimal response for Player One; thus (2.4) with $\mathbf{q}_1 = 1, \mathbf{q}_2 = \mathbf{u}, \gamma = \delta = 0$ gives

$$(\alpha - \beta)\Big(\alpha^2 K(X\mathbf{p} + Y\mathbf{pv}) - \beta^2 K(X\mathbf{pv} + Y\mathbf{pvu})\Big) = 0$$

. This, plus the identical observation with the players reversed, estabilishes full intertwining except when $\alpha = \beta = 1/2$. In that case, $P_1(\mathbf{p}, \mu) = P_1(\mathbf{pu}, \mu)$ so $\mathbf{pu}$ must be optimal; i.e. we can take $\mathbf{v} = \mathbf{u}$. ∎

4. Minimal Payoffs and Opting Out

Theorem 3.3. classifies all mixed strategy Nash equilibria in generic games. Here we briefly address the issue of whether these equilibria survive in a larger game where the players can opt out of the assigned communication protocol.

A key tool is the very simple Theorem 4.1; this and its corollary 4.1A apply to all two by two games (whether generic or not) and are of independent interest:

Theorem 4.1. Let $\mathbf{G}$ be a game with payoff pairs $(X_1, Y_1), \ldots (X_4, Y_4)$. Then in any mixed strategy quantum equilibirum, Player One earns at least $(X_1 + X_2 + X_3 + X_4)/4$.

Proof. Player One maximizes the quadratic form (2.2.1) over the sphere $\mathbf{S}^3$. The trace of this form is $X_1 + X_2 + X_3 + X_4$, so the maximium eigenvalue must be at least $(X_1 + X_2 + X_3 + X_4)/4$. ∎

Corollary 4.1A. If, in Theorem 4.1, the game **G** is zero-sum, then in any mixed strategy quantum equilibrium, Player One earns exactly $(X_1 + X_2 + X_3 + X_4)/4$.

Proof. Apply (4.1) to both players. ∎

4.2. Remarks on Opting Out. A player can throw away his entangled penny and substitute an unentangled penny (or for that matter a purely classical penny, but this offers no additional advantage, because the unentangled quantum penny can always be returned in one of the two classical states **H** or **T**). However, a simple quantum mechanical calculation shows that if Player One unilaterally substitutes an unentangled penny, then no matter what strategies the players follow from there, the result is a uniform distribution over the four possible outcomes. By Theorem 4.1, Player One considers this weakly inferior to any $\mathbf{G}^Q$ equilibrium. Thus, even if we allow players to choose their pennies, all of the $\mathbf{G}^Q$ equilibria survive.

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