This is for students who want an intuitive understanding of Euler's equation

$$e^{ix} = \cos x + i \sin x$$

The first thing to notice is that *you cannot possibly hope to understand why two things are equal unless you understand what those two things mean.* No matter how long you stare at the equation

$$\text{glip} = \text{glop}$$

you will never grasp its meaning unless you know what "glip" means. So the first step toward understanding Euler's equation is to make sure you understand the meaning of the function e^x.

If you *don't* understand what e^x means, you have two options. Option One is to give up. Option Two is to *forget all about Euler's equation for a while*, learn what e^x means, and then start thinking about Euler's equation again.

If you *do* understand what e^x means, chances are excellent that you understand it in (at least) one of four ways: As an abbreviation for a certain power series, as the solution to a certain differential equation, as the inverse of the log function, or as the limit of a sequence. Your understanding of Euler's equation will have to be tailored to your understanding of e^x. So here are the four intuitions you can choose among:

1) **If you think of e^x as a power series**, that is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

then substitute ix for x on both sides, remember that $i^2 = -1$, and you'll see that the even terms of the power series add up to $\cos x$ while the odd terms add up to $i \sin x$. Voila.

2) **If you think of e^x as the function whose derivative is itself** (subject to the initial condition $e^0 = 1$), then you can use the chain rule to write

$$\frac{d}{dx} e^{ix} = i e^{ix}$$

Remember that multiplication by i is a 90 degree rotation, so this equation says that the *tangent* to the curve $x \mapsto e^{ix}$ is everywhere *perpendicular* to that curve. It's not hard to convince yourself that this curve is a circle; that is, it's the same as the curve $x \mapsto \cos x + i \sin x$.

3) **If you think of e^x as inverse to the log function**, then write

$$\log(\cos x + i \sin x) = \int_1^{\cos x + i \sin x} \frac{dw}{w}$$

where the integral is taken counterclockwise around the circle. Writing $w = \cos u + i \sin u$, remember that $1/w = \cos u - i \sin u$ so that $\frac{dw}{w} = idu$ (work it out!). Therefore after changing variables we have

$$\log(\cos x + i \sin x) = \int_0^x idu = ix$$

which is essentially Euler's equation.

4) **If you think of e^x as the limit of a sequence**, i.e.

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

then for large n,

$$e^{ix} \approx \left(1 + \frac{ix}{n}\right)^n$$

If you draw a picture, you'll see that $1 + \frac{ix}{n}$ is very close to $\cos(\frac{x}{n}) + i \sin(\frac{x}{n})$. Raise this to the n^{th} power and you get $\cos x + i \sin x$.

If you understand the function e^x, at least one of the above should give you a clear intuition—after a bit of thought, of course. If you don't understand the function e^x, it's quite premature to ask someone to explain why it's equal to something else.

Appendix

Since i is a rotation through 90 degrees, you should expect i^t to be a rotation through $90t$ degrees. This gives

$$e^{t \log i} = i^t = \cos\left(\frac{\pi t}{2}\right) + i \sin\left(\frac{\pi t}{2}\right)$$

Put $x = \pi t/2$:

$$e^{Ax} = \cos x + i \sin x$$

where $A = 2\log i/\pi$. So even this simple argument already tells you that something *like* Euler's equation has to be true; it's just a matter of computing the value of A (or equivalently of $\log i$).