Chapter 3

HIGHER DIMENSIONAL ISSUES IN TRADE THEORY

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The title of this chapter could be read as an invitation to cover the entire content of this volume at a high level of generality. Acceptance of such a rash invitation would both preclude discussion of anything in depth and impinge on the turf of others. I accordingly limit myself by the ruthless application of two principles. First, I address only problems in which dimensionality is itself central. This rules out topics such as the gains from trade [e.g. Samuelson (1939, 1962), Kemp (1962), Krueger and Sonnenschein (1967), and Dixit and Norman (1980)] which are often discussed in a higher dimensional context even though, or perhaps because, dimensionality does not influence the basic argument. (This chapter will assume without elaboration that free trade confers gains relative to autarky.) Second, the issues posed by dimensionality must be important in their own right: I attempt to scale no mountains of generality simply because they are there.

An excellent example of what I want to include is the "law of comparative advantage". This principle is of the very heart and soul of our field. Yet standard textbook discussions emphasize properties which do not generalize to more than two commodities. Furthermore, the difficulties that additional goods create and the properties that do generalize both well illustrate common consequences of high dimensionality.

My two criteria guarantee that a predominant share of this chapter will concern the fate of the modern, or Heckscher–Ohlin, theory of international trade in higher dimensions. This theory, in its standard two-commodity, two-factor version (see Chapter 1) has dominated international trade theory for over thirty years. But this dominance has long been made uneasy by a widespread suspicion that world commerce does not accord well with the theoretical structure. There are two particular areas of concern. The first stems from the fact that the largest part of world trade involves the exchange of roughly similar products between similar economies, whereas the factor endowment theory – and comparative cost theory generally – teaches us to look to international dissimilarities for the causes of trade. A large part of this actual trade is classified as intraindustry even with significant disaggregation. Thus probing its causes requires a high degree of disaggregation, that is, the explicit consideration of a large number of goods. Scale economies and imperfect competition are also central, so the subject will be reserved for Chapter 7.

The second area of concern stems from the Leontief Paradox [Leontief (1953, 1956)]. Among the huge volume of resulting empirical work (see Chapter 10) the hypotheses that have by and large proved most useful (e.g. human capital, natural resources, skill groups) are not inconsistent with the view that trade has a factor endowments basis, but do demand an increase in dimensionality. Few would now dispute the conclusion of Baldwin (1971, p. 141), "that a straight-forward application of the two-factor (capital and labor) factor-proportions model along Heckscher–Ohlin lines is inadequate for understanding the pattern of U.S. trade."

The sensitivity to higher dimensions of the basic propositions of the modern theory of international trade is the key issue for the practical relevance of the logical structure that has dominated trade theory in the past thirty years.

1. Basic concepts

Before getting down to issues I introduce some notation and concepts that will prove useful. The *national product function* $y(p,V)$ records the maximal income that a country can achieve if facing the vector p of commodity prices and if endowed with the vector V of primary factors. This function therefore depends upon technology and subsumes an optimization process. Just as there is an accounting identity between the total value of national output and the total payment to primary factors, $y(p,V)$ can be given a dual interpretation: the minimal amount paid the factors V, given that factor rewards must be such as to leave the respective costs of production of all commodities no less than the elements of p. Thus

$$y(p,V) = pX = wV, \tag{1.1}$$

where X is the vector of commodity outputs and w the vector of factor rewards. X is chosen to maximize pX subject to the constraint that X be producible from V, and w is chosen to minimize wV subject to the constraint that costs be no less than p.

Differentiating y with respect to any commodity price P_i and using the first definition reflected in (1.1):

$$\frac{\partial y}{\partial P_i} = X_i + \sum_j P_j \frac{\partial X_j}{\partial P_i} = X_i, \tag{1.2}$$

where the totality of terms under the summation sign vanishes as a condition of maximization. Differentiating y with respect to any factor endowment V_j and using the second definition reflected in (1.1):

$$\frac{\partial y}{\partial V_j} = w_j + \sum_i V_i \frac{\partial w_i}{\partial V_j} = w_j, \tag{1.3}$$

where the summation vanishes as a condition of minimization. Finally, differentiating (1.2) with respect to V_j, (1.3) with respect to P_i, and noting that $\partial^2 y/\partial P_i \partial V_j$

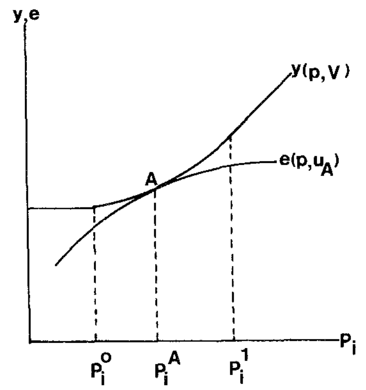

Figure 1.1. Autarkic equilibrium.

$= \partial^2 y/\partial V_j\, \partial P_i$, gives a set of what are called "reciprocity conditions":

$$\frac{\partial X_i}{\partial V_j} = \frac{\partial w_j}{\partial P_i}. \tag{1.4}$$

Discussion this far has been confined to the supply side of an economy. To facilitate summary of the demand side, assume a single collective utility function over national consumption (abandonment of this assumption will not be an objective of this chapter). The *national expenditure function* $e(p,u)$ records the minimum that must be spent at commodity prices p to purchase a consumption bundle yielding utility no less than u. Write

$$e(p,u) = pD, \tag{1.5}$$

where D denotes the chosen bundle. Then differentiation of (1.5) with respect to any commodity price P_i gives:

$$\frac{\partial e}{\partial P_i} = D_i + \sum_j P_j \frac{\partial D_j}{\partial P_i} = D_i, \tag{1.6}$$

where the summation vanishes as an optimization condition.[1]

Figure 1.1 depicts autarkic equilibrium. P_i is measured along the horizontal axis, and all other commodity prices are presumed set equal to their equilibrium values. The slopes of the income and expenditure functions are respectively X_i

[1]For surveys of duality theory see Diewert (1974, 1978). Summaries and applications to international trade are contained in Dixit and Norman (1980) and Woodland (1982). See also Samuelson's classic (1953) and Appendix One to Ethier (1983).

and D_i, from (1.2) and (1.6) so that point A reflects the autarkic equilibrium condition $X_i = D_i$, and P_i^{A} is the equilibrium price.

The illustrated curvatures of the income and expenditure functions follow from the respective subsumed optimizations. The position of $y(p, V)$ depends of course on the endowment vector V, and autarkic utility u_{A} must be such as to position $e(p, u_{\mathrm{A}})$ tangent to the income function as shown. P_i^0 denotes the price at which the economy ceases to produce good i – given the values of all other commodity prices – and P_i^1 the price at which the economy specializes completely in X_i.

2. The law of comparative advantage

In two dimensions the law of comparative advantage – that a comparison of home and foreign relative autarkic prices predicts the pattern of trade and gives bounds for the terms of trade – is rather robust across models, even though it is widely appreciated that the proposition can be vitiated by certain phenomena, such as multiple autarkic equilibria or scale economies. Ignore such possibilities so as to focus clearly on the effects of dimensionality.

Figure 2.1 shows a movement from autarky to mutual free trade for a pair of countries, when there are only two goods. This movement can be depicted as an upward shift of the expenditure function since free trade gives higher utility than autarky. Points T and T^* denote the respective trade equilibria (throughout this essay I use an asterisk to refer to the rest of the world), with common price P_i^{T} between the respective autarkic prices. At T, y is flatter than e so that $D_i > X_i$

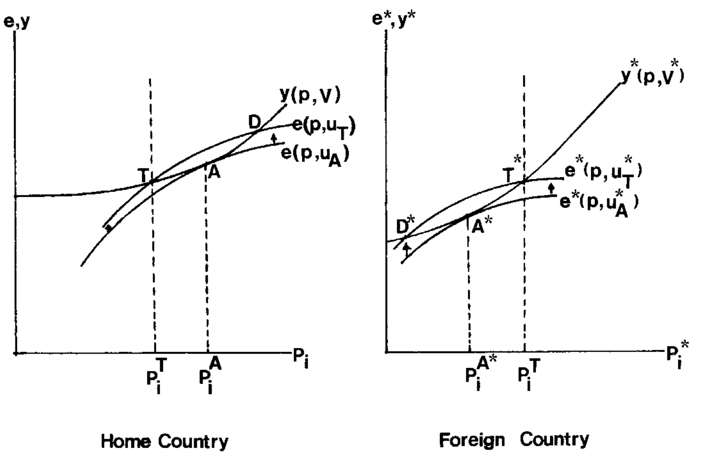

Figure 2.1. The law of comparative advantage.

from (1.2) and (1.6) and similarly $X_i^* > D_i^*$ at T^*. Thus the home country imports good i, as predicted by a comparison of autarkic prices.[2]

Is Figure 2.1 an adequate description of the movement to free trade? With only two goods it is, because then the figure's implicit assumption that all other commodity prices are fixed amounts only to a choice of numeraire: P_i^{T} denotes the terms of trade, and the home country exports the other good in exchange for good i. The figure is also an adequate analysis in higher dimensions if the movement to free trade involves what can be called a *two-dimensional* price change: the prices of a group of goods all move equiproportionally relative to the prices of all other goods. For then the analysis can be based upon two composite commodities. I mention this because the extent to which attempted generalizations of two dimensional results in fact apply to more interesting situations will sometimes be an issue in what follows.

When the movement to free trade does involve a change in more than one relative price – that is, when one goes beyond two goods in a substantive way – Figure 2.1 becomes inadequate. For changes in relative prices, other than that of good i in terms of the numeraire, will produce shifts of the income and expenditure functions. A little experimenting with pencil and paper convinces one that practically anything can apparently be made to happen as regards both the trade pattern and the magnitude of P_i^{T}.

Thus higher dimensional generalization requires restrictions of some sort. A natural point of departure is to ask when the 2×2 results remain fully valid in some respect, say as regards the pattern of trade. That is, when does it remain true, in a multidimensional context, that 2×2 comparisons determine the pattern of trade? When will the pairwise comparison,

$$\frac{P_i^{\mathrm{A}}}{P_j^{\mathrm{A}}} > \frac{P_i^{\mathrm{A}*}}{P_j^{\mathrm{A}*}}, \tag{2.1}$$

necessarily imply, by itself, that in free trade the home country will export good j to the rest of the world in exchange for good i? A little reflection answers, "never". The problem comes from "intervening" goods, which necessarily present themselves whenever there are more than two commodities. To grasp the point immediately, note that (2.1) is equivalent to

$$\frac{P_i^{\mathrm{A}}}{P_i^{\mathrm{A}*}} > \frac{P_j^{\mathrm{A}}}{P_j^{\mathrm{A}*}}.$$

[2] The free trade equilibrium cannot be shown by points D and T^* because this would require both countries to export good i. Likewise the pair D^* and T would require both countries to import the good. Finally, the pair D, D^* is ruled out because it would imply $P_i^{\mathrm{T}} > P_i^{\mathrm{A}} > P_i^{\mathrm{A}*} > P_i^{\mathrm{T}}$.

Suppose there are three distinct goods and index them so that

$$\frac{P_i^{\mathrm{A}}}{P_i^{\mathrm{A}*}} > \frac{P_k^{\mathrm{A}}}{P_k^{\mathrm{A}*}} > \frac{P_j^{\mathrm{A}}}{P_j^{\mathrm{A}*}}.$$

A pairwise comparison of goods i and k indicates that the home country exports k (and imports i) whereas a pairwise comparison of k and j yields the contrary conclusion that the home country imports k (and exports j). The important point about this example is that it necessarily results as a consequence solely of an increase in dimensionality beyond two goods: *no* restrictions on technology or preferences can set matters aright (unless they undo the increase in dimensionality by implying that all price changes are two dimensional).

Given this insuperable difficulty, one asks if some modified version of the 2×2 proposition might generalize. A natural candidate would be a *chain* version: number the n goods so that

$$\frac{P_1^{\mathrm{A}}}{P_1^{\mathrm{A}*}} > \frac{P_2^{\mathrm{A}}}{P_2^{\mathrm{A}*}} > \cdots > \frac{P_n^{\mathrm{A}}}{P_n^{\mathrm{A}*}}. \tag{2.2}$$

Might it not be true that the free trade equilibrium would break the chain somewhere, with all goods with price ratios strictly to the left of the break imported by the home country and all those strictly to the right exported? Such a proposition subtly alters the conclusions that would be drawn from a pairwise comparison such as (2.1). Instead of saying "the home country will import good i and export good j", we now make the *conditional* prediction that "the home country will import good i if it imports good j and will export good j if it exports good i".

Such an approach is supported by the fact that chains like (2.2) do confer such predictions in certain cases. For example, in a Ricardian model, where an arbitrary number of goods are allowed but there is only one primary factor, the terms in (2.2) coincide with relative labor requirements so that, when the ratio of the domestic free trade wage to the foreign is known, the home country must import all those goods corresponding to terms in (2.2) strictly greater than this ratio and export all those goods strictly less.[3] But such a result is not generally available.

[3]See Haberler (1936). The case allows arbitrary dimensionality concerning goods, but restricts the factors to one and the countries to two. This essay will not have much to say about the consequences of additional countries, because the problems they raise are usually straightforward and sometimes tedious. But occasional exceptions will be noted. This is one. The case of two goods, one factor, and many countries yields a chain analysis analogous to the above. But allowing both many goods and

This is because in the general environment a chain of price comparisons does not translate directly into a chain of output comparisons or of demand comparisons, a fact well appreciated by students of microeconomics.[4] Suppose that the price vectors p^A and p^{A*} in (2.1) and (2.2) refer not to two countries, but to two distinct equilibria (autarky and free trade, for example) for a single country with a given endowment vector. With only two commodities, one can expand only by drawing resources from the other; (2.1) does indeed imply:

$$\frac{X_i}{X_j} \geq \frac{X_i^*}{X_j^*},$$

if i and j are the only goods. But (2.2) does not imply:

$$\frac{X_1}{X_1^*} \geq \frac{X_2}{X_2^*} \geq \frac{X_3}{X_3^*} \geq \cdots \geq \frac{X_n}{X_n^*}.$$

Suppose that X_1 and X_3 make relatively intensive use of two disjoint groups of factors and that X_2 relies heavily on both groups. Then an expansion of X_1 that draws resources from X_2 would also free factors allowing X_3 to expand: X_3 might well rise relative to X_2 even though P_3 falls relative to P_2. Note that the existence of at least three goods is essential to this example.

On the demand side, price variations will produce changes in real incomes and these will affect demands in ways unrelated to the price change. But such possibilities arise even in two dimensions: the Giffen Paradox has graced undergraduate texts for years. What additional goods allow is complementarity between goods on the demand side, analogous to the above illustration, in response to compensated price changes.

But even though detailed predictions are dangerous, it is not true that *anything* can happen. If a rise in P_1 causes an increase in X_1 the necessary resources must come from somewhere; although X_3 may be complementary to X_1, the rest of the economy, in some average sense, cannot be. This idea can be made precise as follows. Let X^0 and X^1 be the output vectors produced with prices p^0 and p^1 by an economy with endowment V. Then $p^0X^0 = y(p^0, V) \geq p^0X^1$ since X^0 maximizes income at p^0 even though X^1 is feasible. Similarly $p^1X^1 \geq p^1X^0$. The two

many countries introduces problems of its own. The assignment of goods to countries to produce them that will permit the world to obtain an efficient output vector obviously depends upon the production techniques of all goods in all countries and so cannot be exposed by any sort of chain of bilateral comparisons. See McKenzie (1954, 1956), Jones (1961), and also Wilson (1980).

[4]For a discussion in the context of international trade, see Drabicki and Takayama (1979) and Dixit and Norman (1980, p. 94–96). Both provide counterexamples to the straightforward application of 2×2 results to higher dimensions.

inequalities together yield:

$$(p^0 - p^1)(X^0 - X^1) \geq 0. \tag{2.3}$$

That is, output changes must be *positively correlated* with price changes, so that the latter predict the former in an average sense.

In like fashion, if D^0 and D^1 are demanded at prices p^0 and p^1 when utility is held constant:

$$(p^0 - p^1)(D^0 - D^1) \leq 0. \tag{2.4}$$

Demand changes are negatively correlated with compensated price changes.

Note two points. First, when there are only two goods (2.3) and (2.4) imply the usual unambiguous responses. Second, (2.3) and (2.4) are quite general, especially as regards dimension, and followed directly from the optimization subsumed in the product and expenditure functions. A number of standard 2×2 results generalize along these lines.

One would certainly expect this to be true of the law of comparative advantage, since it attempts to link trade to price divergences and since trade is but the difference between production and consumption. To proceed,[5] note that, if p^{A} denotes the autarkic price vector:

$$p^{\mathrm{A}}D^{\mathrm{T}} \geq e(p^{\mathrm{A}}, u_A) = y(p^{\mathrm{A}}, V) \geq p^{\mathrm{A}}X^{\mathrm{T}},$$

since the free trade consumption vector D^{T} yields at least u_{A} of utility, and the free trade production vector X^{T} can be produced from V. Then $p^{\mathrm{A}}M \geq 0$ where $M = D^{\mathrm{T}} - X^{\mathrm{T}}$ denotes net imports. Balanced trade requires $p^{\mathrm{T}}M = 0$ so that

$$(p^{\mathrm{A}} - p^{\mathrm{T}})M \geq 0. \tag{2.5}$$

Imports are positively correlated with the excesses of autarkic prices over free trade prices. A similar argument yields $(p^{\mathrm{A}*} - p^{\mathrm{T}})M^* \geq 0$ for the foreign country, and $M = -M^*$. Thus

$$(p^{\mathrm{A}} - p^{\mathrm{A}*})M \geq 0. \tag{2.6}$$

Thus autarkic price differences do indeed predict trade patterns in the average sense of a positive correlation between the two. Finally, apply (2.3) to a comparison of free trade and autarkic equilibria:

$$(p^{\mathrm{A}} - p^{\mathrm{T}})(X^{\mathrm{A}} - X^{\mathrm{T}}) \geq 0. \tag{2.7}$$

[5]See Deardorff (1980) and Dixit and Norman (1980, pp. 94–96), and also Appendix One of Ethier (1983).

On average, trade causes countries to redirect resources away from those sectors with lower prices than in autarky.

3. The basic propositions of the modern theory

The 2×2 Heckscher–Ohlin–Samuelson model yields four central results: the factor-price equalization, Stolper–Samuelson, Rybczynski, and Heckscher–Ohlin theorems. To establish a frame of reference I state eight propositions reflecting the principal variants of these theorems. See Chapter 1 for a fuller treatment.

Proposition 1 (Stolper – Samuelson)

A small change in relative prices will increase, in terms of both goods, the reward of the factor used intensively in the production of that good whose price has risen and will reduce, in terms of both goods, the reward of the other factor, provided that both goods are produced.

Note that there are two aspects to this proposition. First, one factor reward rises in real terms and one falls, independently of how recipients of those rewards spend them, so that commodity price changes generate conflict. Second, the identities of the favored and punished factors can be determined by relative factor intensities.

Proposition 2 (global Stolper – Samuelson)

Proposition 1 applies to large price changes as well, provided that endowments are held fixed or that the technology does not exhibit factor intensity reversals.

Proposition 3 (factor-price equalization)

For each relative commodity price there exists a cone of endowments such that all countries in the cone, and with the given technology, will have identical factor prices when freely trading at those world prices. The cone is non-trivial as long as it does not coincide with a factor-intensity reversal.

Proposition 4 (global univalence)

If there are no factor-intensity reversals, any two countries with the same technology must have equal prices if freely trading at a common world price and if both countries diversify.

Note that Proposition 3 in effect says that factor-price equalization results if the two countries have "sufficiently similar" factor endowments: widely divergent endowments preclude equalization regardless of the global nature of the technology. Proposition 4 imposes a global property to make equalization equivalent to diversification in production.

Proposition 5 (Rybczynski)

At constant relative commodity prices, a small change in factor endowments will increase, relative to both factors, the output of the good making intensive use of the factor which has become relatively more abundant and will reduce the output of the other good relative to both factors, if the economy is diversified.

Note that this proposition has two distinct aspects as did Proposition 1, and that the two propositions are in a sense dual.

Proposition 6 (global Rybczynski)

Proposition 5 applies as well to any large changes which do not disturb diversification, if the technology has no factor-intensity reversals.

Proposition 7 (quantity version of the Heckscher–Ohlin theorem)

Suppose two countries have identical technologies with no factor intensity reversals and identical homothetic demands. Then in free trade each country will export the good making relatively intensive use of the country's relatively abundant factor.

Proposition 8 (price version of the Heckscher–Ohlin theorem)

Suppose two countries have identical technologies with no factor-intensity reversals. Then each country has a lower (compared to the other country) relative autarkic price of the good making relatively intensive use of the factor which would be relatively cheap in that country in autarky. Also that good would be exported in free trade if autarkic equilibrium is unique in each country.

4. Many goods

As Jones and Scheinkman (1977) have pointed out, the 2×2 model is special in two ways: the dimensionality is low and the number of goods exactly equals the number of factors. To disentangle the individual implications, I first examine cases where the number of goods alone, and then the number of factors alone, is allowed to exceed two.

Suppose then the conventional Heckscher–Ohlin framework with the sole exception that the number of goods is arbitrary but greater than two. Equilibrium requires that (4.1), (4.2) and (4.3) hold for each country:

$$p \leq wA(w), \tag{4.1}$$

$$[p - wA(w)]X = 0, \tag{4.2}$$

$$A(w)X = V. \tag{4.3}$$

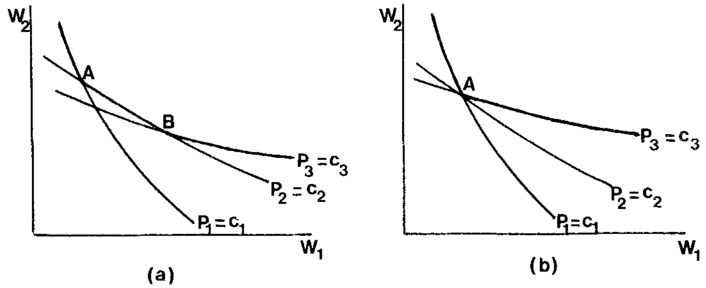

Figure 4.1. Equilibria with three goods and two factors.

In these expressions, p denotes the vector of n commodity prices, w the (two-dimensional) vector of factor rewards, X the n-vector of commodity outputs and V the vector of (two) factor endowments. The matrix $A(w)$ is the array of least-cost techniques at factor rewards w, so that $wA(w)$ is the vector of unit cost functions, $c(w)$.

4.1. *Factor-price equalization*

A considerable literature has concerned the question of whether additional goods render factor-price equalization more or less likely than in a 2×2 environment.[6] The sensitivity of any measure of "likelihood" to its frame of reference renders the question too sterile to be of much inherent interest. Nevertheless it has exposed the essential features of the $n \times 2$ context.

Suppose first that some country, engaged in free trade, produces positive amounts of at least two goods, say X_1 and X_2. Then the first two inequalities of (4.1) must in fact be strict equalities, and this subsystem can be analyzed in the normal 2×2 way. In particular Proposition 4 (global univalence) holds with regard to any trading partner also producing X_1 and X_2. Thus if any pair of goods is free of factor-intensity reversals, factor-price equalization must characterize free trade between any two countries both producing those two goods; if the technology has no pair-wise factor-intensity reversals at all, any countries producing at least two goods in common must have equal factor prices if freely trading. Looked at in this light, increasing the number of goods appears to broaden the opportunity for factor-price equalization.

But different viewpoints yield different interpretations. Figure 4.1 shows isocost curves for goods 1 and 2. Each curve depicts the collection of factor prices w_1 and

[6]References include Samuelson (1953), Tinbergen (1949), Meade (1950), Land (1959), Johnson (1967, 1970), Melvin (1968), Bertrand (1970), Chang (1979), and Dixit and Norman (1980).

w_2 that cause the minimum unit cost of the respective good to equal a specified price; the isocost curve of good i is the graph of solutions to

$$P_i = c_i(w_1, w_2) = w_1 a_{1i}(w_1, w_2) + w_2 a_{2i}(w_1, w_2), \tag{4.4}$$

for given P_i, where a_{ij} denote elements of A. (See Chapter 1 for a fuller description.) Any $w_1 - w_2$ combination lying outside an isocost curve implies a cost for the respective good greater than its price, so the good cannot continue to be produced in equilibrium. Likewise, points below the curve imply a minimum cost less than the price of the product and are therefore inconsistent with long-run equilibrium. Thus the latter requires that factor prices be indicated somewhere along the outer envelope – called the "factor price frontier" – of all isocost curves corresponding to the actual commodity prices. (Values of the national product function $y(p, V)$ can be thought of as determined by the process of minimizing wV over all w not below the factor-price frontier determined by p.)

Suppose that in equilibrium goods 1 and 2 are produced in positive amounts. Then, with the given prices of these two goods, factor prices must be as indicated by the intersection point A in Figure 4.1(a). This equilibrium will allow good 3 to be produced also only if P_3 happens to have just the right value for its isocost curve to pass through A, that is, if the cost of producing X_3 implied by the factor prices indicated by A equals P_3, as in Figure 4.1(b). If there are n goods, complete diversification requires a common intersection for all n isocost curves.

Suppose that the three commodity prices are not just such as to yield a common intersection, as in Figure 4.1(a). Suppose that all goods are produced somewhere in the world (since we are, after all, concerned with such a case) and let there be two countries – the home economy and the rest of the world. Suppose X_1 is produced at home. Then home factor prices must be indicated by some point on the factor-price frontier in Figure 4.1(a) at or above point A. Since X_3 is necessarily produced in the rest of the world, its factor price must be reflected by a point at or below B. Neither country can be on the segment AB, exclusive of the endpoints. Thus the two countries cannot possibly have identical factor prices, and those at home must differ from those abroad by at least the distance AB. Only if the prices of the three goods happen to be such as to yield Figure 4.1(b) is factor price equalization possible, when it must occur at A if both countries completely diversify.

This argument seems to reduce factor-price equalization to a fluke. And so it would if commodity prices were drawn from an urn. But they are not: they are determined so as to clear world commodity markets.

To see what this implies note first that, from (4.4):

$$\mathrm{d}P_i = [(\mathrm{d}w_1)a_{1i} + (\mathrm{d}w_2)a_{2i}] + [w_1(\mathrm{d}a_{1i}) + w_2(\mathrm{d}a_{2i})]. \tag{4.5}$$

Now the second term in brackets on the right hand side must equal zero, as a necessary condition for cost minimization, and so the first bracketed term must likewise vanish for a movement along an isocost curve, where $\mathrm{d}P_i = 0$. Thus $(\mathrm{d}w_2/\mathrm{d}w_1) = -(a_{1i}/a_{2i})$: the slope of an isocost curve at any point equals (minus) the relative factor proportions employed in the respective industry at the relevant factor price.

A country's relative factor endowment is necessarily a weighted average of the factor proportions it employs in its operating sectors. Thus if the home economy has factor prices at A or above on the factor price frontier of Figure 4.1(a), the home relative endowment V_1/V_2 necessarily exceeds the slope of the P_3 curve at A. Likewise, for the foreign economy to be at B or below requires that the foreign relative factor endowment be less than the slope of P_3 at point B. If these conditions are not met – for example if relative endowments in the two countries are more nearly equal than the slopes of the P_3 curve at A and B – the commodity prices which give the isocost curves the position indicated by Figure 4.1(a) could not possibly clear world commodity markets.

For both countries to be at point A in Figure 4.1(b) both relative factor endowments must lie between the slopes of the P_1 and P_3 isocost curves at A. If this is not the case – for example because endowments are too dissimilar – factor prices cannot be equalized.

Then when account is taken of commodity market equilibrium the picture that emerges is qualitatively similar to that in the 2×2 case: trade between countries with "sufficiently similar" relative factor endowments will produce factor price equalization and "sufficiently dissimilar" endowments preclude such equalization.

4.2. Stolper – Samuelson

From (4.5) we obtain, for any good actually produced,

$$\hat{P}_i = \theta_{1i}\hat{w}_1 + \theta_{2i}\hat{w}_2, \tag{4.6}$$

where a circumflex denotes proportional change (so that $\hat{P}_i = (\mathrm{d}P_i)/P_i$, etc.), and $\theta_{1i} = w_1 a_{1i}/P_i$, factor 1's distributive share in sector i (so that $\theta_{1i} + \theta_{2i} = 1$). Thus the proportional change in the price of any produced commodity is a weighted average of the proportional changes in the two factor rewards. The presence of additional produced goods does not disturb the 2×2 Stolper–Samuelson logic (see Chapter 1). Any change in the relative prices of produced goods necessarily raises one factor reward in terms of all (still) produced goods and lowers the other factor reward in terms of all (still) produced goods, and the identification of the respective factor can be deduced from the relative factor intensities of *any* pair of produced goods whose relative price changes.

But we have seen that, unless relative factor endowments are sufficiently similar across countries, some countries will not produce all goods. If more goods than factors are initially produced, a fall in any single commodity price will in fact cause some good to cease being produced.[7] If the prices of all non-produced goods fall, say, relative to produced goods, and if the latter prices do not change relative to each other, all factor rewards rise in terms of non-produced goods and remain unchanged in terms of produced goods. This is obviously analogous to the case of specialization in a 2×2 context. But different consequences arise when: (a) some non-produced goods rise in price relative to produced goods and some fall, and/or (b) the prices of produced goods change relative to each other and relative to some non-produced goods' prices. In such cases the rewards of some factors could rise relative to some goods and fall relative to other goods, a possibility that is absent when there are only two commodities. Nevertheless it is fair to conclude that the Stolper–Samuelson theorem is not fundamentally altered by an increase in the number of goods.

4.3. *Rybczynski*

Suppose that the home country is producing more than two goods, that is, that for a given V and an equilibrium w, eq. (4.3) possesses a solution X with at least three positive components. Then there must be many such solutions, because (4.3) is a system of two linear equations in n unknowns and will accordingly be satisfied by an $n-2$ dimensional hyperplane of X vectors. Thus without further information national outputs are indeterminate whenever world prices and the national factor endowment permit positive production of more than two goods.[8]

Consider a change in factor endowments at unchanged commodity prices. The discussion of factor-price equalization justifies supposing that factor prices also remain unaltered. But with X indeterminate both before and after the change in V, so are the resulting changes in outputs. Thus increasing the number of goods above two profoundly affects the Rybczynski theorem.

4.4. *The Heckscher–Ohlin theorem*[9]

Suppose that the home and foreign economies each produce at least two goods in common. Then their factor prices must be equal so that they share a common

[7]Expression (4.6) and its accompanying logic can be retained even for goods which cease to be produced (or which initially become produced) if the left-hand side is reinterpreted as the change in supply price rather than market price. This is analogus to the interpretation made in Section 5 below for the Rybczynski theorem when factors outnumber goods. (I owe this observation to Avinash Dixit.)

[8]This indeterminancy of output when goods outnumber factors has often been emphasized in the literature. See Samuelson (1953), Travis (1964, 1972), Melvin (1968), Kemp (1969), Pearce (1970), Vanek and Bertrand (1971), Chang (1979), and Dixit and Norman (1980).

[9]Useful references include Jones (1956, 1974), Bhagwati (1972), Krueger (1977), Deardorff (1979), and Dornbusch, Fischer and Samuelson (1980).

technology matrix $A(w)$. (I ignore the possibility of factor-intensity reversals, since they can vitiate the Heckscher–Ohlin theorem even in a 2×2 world.) This means that the domestic hyperplane of possible outputs, defined by (4.3), is parallel to the analogous foreign hyperplane. Then the equilibrium vector of world outputs $X + X^*$ can be attained by many combinations of X and X^* consistent with (4.3) and its foreign analog. Commodity trade flows are indeterminate and cannot be predicted by any theory. Indeed the *pattern* of commodity trade need not be determinate.

If factor prices are not equalized the situation becomes more clearcut. Number the n goods in order of their relative factor intensities, good 1 making the most intensive use of factor 1. (Continue to assume away factor-intensity reversals so that this ranking is determinate.) Then the factor price frontier corresponding to the equilibrium commodity prices must consist of segments of the isocost curves for goods 1, 2, 3, etc. respectively, going downwards and to the right, as in Figure 4.2 for the case $n = 5$. (Some of the segments could be single points, if three or more isocost curves have a common intersection as at point B in Figure 4.2.) The successive segments become flatter because of the convexity of the individual isocost curves. If the home country is relatively most abundant in factor one, that is, has the highest V_1/V_2 ratio, its factor prices must be on the steeper part of the factor-price frontier: the home economy must export those goods "higher up" in the chain of commodities and import the others. In Figure 4.2, good 1 is exported, goods 3, 4, and 5 are imported, and good 2 is in an ambiguous intermediate position, if the home economy is at A. Note that Figure 4.2, as drawn, requires at least three countries for all five goods to be produced.

If there are many countries they can also be ordered on the basis of their V_1/V_2 ratios, and to each country there will correspond a segment of the commodity chain having the property that the country will export the goods in the segment and import the other goods. Again, "borderline" goods, producible in more than one country and of indeterminate trade status, may exist.

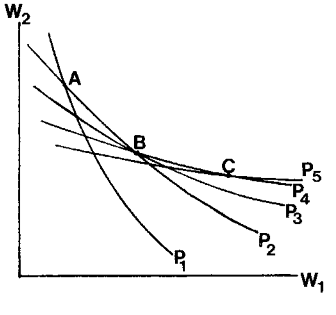

Figure 4.2. Many goods.

In the two country case, if two goods are producible in both countries factor prices are equalized and so all goods are producible in both countries: the chain proposition says nothing. With many countries, factor-price equalization between some subset of countries does not imply that all goods are producible by the members of that subset. The chain proposition still holds if the countries in the subset are treated as a single joint unit. Suppose for example that one country has an equilibrium at A in Figure 4.2, another is at B, and two others are at C. The factor prices of the latter two countries are equalized, with X_4 and X_5 producible in both of them. The two countries together import X_1, X_2, and X_3 and export X_5; X_4 is an ambiguous "borderline" good. This trade pattern need not hold for each of the two subgroup countries individually: one of them may export X_5 to the other, for example.

Factor intensity reversals prevent the construction of a unique commodity chain. However, in the two country case it is easy to see that the chain proposition still holds in a descriptive sense, when the chain is constructed on the basis of the *actual* factor intensity of each good in the country of export. (In the many country case statement of the proposition becomes cumbersome because a commodity might be exported by more than one country.)

5. Many factors

5.1. The basic propositions

The presence of many commodities influences but by no means disembowels the basic theory. The factor-price equalization and Stolper–Samuelson properties are basically intact, the Rybczynski and Heckscher–Ohlin theorems more significantly affected. Analytically the basic modification is that, for given factor prices and endowments, the set of equations (4.3) leaves commodity outputs X indeterminate.

Suppose now that the number of goods returns to two but that there are $m > 2$ factors of production.[10] If both goods are produced, with given commodity prices, the system (4.1) becomes a set of 2 equations in n unknowns: factor prices are not in general determined solely by commodity prices but also depend upon other information, notably factor endowments. Consider a free-trade equilibrium in which foreign endowments differ only very slightly from home endowments, by the vector $\mathrm{d}V$. Then if foreign factor prices are to equal those at home, w, it follows from (4.3) that the difference $\mathrm{d}X$ between home and foreign outputs must

[10]The implications of more factors than goods are studied in Samuelson (1953), Jones (1979, ch. 8), Diewert and Woodland (1972), Batra and Casas (1976), Jones and Easton (1983), and Egawa (1978).

satisfy:

$$A(w)(\mathrm{d}X) = \mathrm{d}V. \tag{5.1}$$

Now the two-dimensional vector $\mathrm{d}X$ will in general be determined by the first two equations of (5.1) so that, except for a fluke, the remaining $m-2$ equations cannot be satisfied: factor prices will have to differ between countries.

When the same logic that has just been used for a comparison between two countries is instead applied to a comparative static change in a single country, one concludes that a change in endowments, at constant commodity prices, will produce a change in factor prices and a consequent shift in production techniques. The basis for the logic of the Rybczynski theorem is then destroyed. The proposition can be resuscitated if sticky factor rewards are maintained as an *assumption* and if factor markets are no longer required to clear, and $\mathrm{d}V$ in (5.1) is interpreted as the vector of factor demands. Then (5.1) gives a set of equations of the form:

$$\lambda_{j1}\hat{X}_1 + \lambda_{j2}\hat{X}_2 = \hat{V}_j, \qquad j = 1, \ldots, m, \tag{5.2}$$

where $\lambda_{ji} = a_{ji}X_i/V_j$ denotes the fraction of aggregate demand for factor j contributed by sector i. Thus every $\hat{V}_j$ is a weighted average of $\hat{X}_1$ and $\hat{X}_2$, so that any change in relative factor demands must be accompanied by an increase in the output of one good relative to all factor demands and a reduction in the other commodity output relative to all factor demands. The Rybczynski result is preserved when the number of factors rises in the same way that the Stolper–Samuelson result is preserved when the number of goods rises. But preservation of the former requires a drastic alteration in the circumstances under which it applies.

Finally, note that, with both goods produced, (4.1) leads to

$$\hat{P}_i = \theta_{1i}\hat{w}_1 + \cdots + \theta_{mi}\hat{w}_m; \qquad i = 1, 2. \tag{5.3}$$

Each $\hat{P}_i$ is a weighted average of all $\hat{w}_j$. A moment's reflection reveals that this is quite consistent with some of the $\hat{w}_j$ being weighted averages of $\hat{P}_1$ and $\hat{P}_2$: a change in relative commodity prices might well cause some factor rewards to increase in terms of one good while falling in terms of the other.

5.2. *The specific-factors model*

The complications introduced by more than two factors are well illustrated by the specific-factors model, sufficiently prominent in recent years to deserve mention in its own right.[11] This structure differs from the standard 2×2 one in that one of the factors is immobile between industries, so that its two sectoral allocations are distinct specific factors. Thus the model is 2×3 though each good is still produced by only two factors. Consider an increase in the endowment of one of the specific factors, with commodity prices constant. Constant factor prices would require unchanged techniques in both industries, but the sector with more of the specific factor can maintain its original factor proportions only by attracting some of the mobile factor from the other industry, thereby changing factor proportions there: the failure of factor-price equalization is transparent here.[12] An increased endowment of the mobile factor also must change techniques, with that factor allocated to the two sectors so as to preserve equality between the values of its marginal products. This produces the anti-Rybczynski result that both outputs rise but proportionally less than the endowment of the mobile factor. Finally, changes in relative commodity prices always raise the reward of the mobile factor in terms of one good and lower it in terms of the other. That factor will move towards the sector with the increased price; this lowers the value of the mobile factor's marginal product in terms of that good, while the exit of that factor from the other sector raises the value of its marginal product in terms of that other good.

6. Strong results in even technologies

Standard 2×2 conclusions are affected by increases in either the number of goods alone or the number of factors alone, with the latter the more devastating. I turn now to "even" increases in dimensionality: $n = m > 2$. The general $n \times m$ case can be thought of as a composite of the odd conclusions just derived and the even ones to come. But I shall leave to the reader the actual task of composition, except when there is reason to discuss it explicitly.[13]

Investigations of the $n \times m$ case have followed two distinct approaches. The first, to be discussed in this section, has formulated general propositions thought to retain as much as possible of familiar 2×2 properties and then derived necessary conditions for general validity. Examination of these conditions then

[11]For details of this model see Samuelson (1971), Jones (1971), Mayer (1974), Mussa (1974), and Neary (1978).

[12]Samuelson (1971) discusses the tendency of free trade to partially equalize factor prices between countries.

[13]This chapter does not make as intensive use of matrix algebra as has become customary in its field. For details of the mathematical structure of the general production model see, in addition to Samuelson (1953) of course, several recent valuable contributions with clear expositions and extensions: Diewert and Woodland (1977), Jones and Scheinkman (1977), Chang (1979), and Takayama (1981).

sheds light on the practical relevance of the "strong" properties of the 2×2 world.

An alternative strategy, taken up in the next section, is to ask what results remain valid in an $n \times m$ context in the absence of any assumptions stronger than those commonly imposed in a 2×2 context.

6.1. Global univalence

Perhaps most attention has been lavished on the circumstances that allow us to deduce factor price equalization from diversification in production.[14] Evidently an $n \times n$ analog to Proposition 4 would require internationally identical technology and would interpret diversification as the positive production of all n goods. Then the implied problem, in which interest was generated by Samuelson (1953), is to find technological restrictions such that the relation

$$p = wA(w) \tag{6.1}$$

yields a one-to-one mapping between w and p. In two dimensions this follows if there are no factor-intensity reversals, that is, if $A(w)$ is non-singular for all possible w. In higher dimensions the obvious analog to an absence of factor-intensity reversals is the non-singularity of $A(w)$. This condition guarantees the *local* univalence of (6.1): if $A(w)$ is non-singular for $w = w_0$, there exists some neighborhood of w_0 over which (6.1) is one-to-one. But when $n > 2$, *global* univalence of (6.1) is not guaranteed even if local univalence holds for *all* possible w, in sharp contrast to the $n = 2$ case described in Proposition 4. The distinction is described schematically in Figure 6.1. The set D is mapped into D'. Imagine the mapping as physically "bending" D so as to place point w^2 over w^0.

There is local univalence between a neighborhood of w^0 in D and a corresponding neighborhood of p^0 in D', and also a local univalence between a neighborhood of w^2 in D and one of p^0 in D'. But global univalence fails because both w^0 and w^2 are mapped into p^0.

Though complete local univalence does not imply global univalence in general one might ask whether it could do so once explicit account is taken of mathematical properties additional to (6.1) motivated by economic concerns. For example, $A(w)$ results from the choice of least-cost techniques over a technology of conventional properties, and only non-negative prices are commonly considered of interest. However, a counterexample presented by McKenzie (1967) took the wind out of the sails of such an approach.

[14]Besides the works referred to below, mention should be made of Samuelson (1949, 1967), McKenzie (1955), Harrod (1958), Pearce and James (1951), Pearce (1959, 1967, 1970), and Kuga (1972).

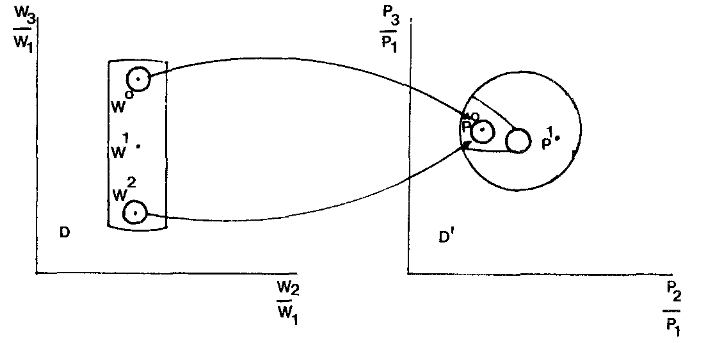

Figure 6.1. A failure of global univalence.

The basic result is that of Gale and Nikaido (1965): the system (6.1) is globally univalent if $A(w)$ always has all positive principal minors (in which case A is sometimes called a "P-matrix"). When $n=2$ this condition reduces (with the appropriate numbering of goods and factors) to the absence of a factor-intensity reversal. Uekawa (1971) has furnished the following interpretation for the general case.

Goods and factors can be numbered so that, no matter how the index set $\{1,\ldots,n\}$ is divided into two proper subsets J and $\bar{J}$, it is always possible to find some positive output vector X for which the goods in J together use more of *each* factor in J and less of *each* factor in $\bar{J}$ than do the goods in $\bar{J}$ together. Thus Uekawa has established a sense in which it can be said that A is a P matrix if and only if the technology resembles that of a 2×2 model.

But the Gale–Nikaido condition is only a sufficient one for global univalence which does not make use of economically reasonable restrictions. Thus the hunt is still on for weaker conditions. Mas-Colell (1979a, 1979b) has shown that positivity of the principal minors of $A(w)$ – other than the determinant of $A(w)$ itself – can be dispensed with whenever w is strictly positive. Following an earlier lead of Samuelson (1966) and Nikaido (1972), Mas-Colell (1979b) has also shown that the cost function (4.1) between *strictly* positive w and *strictly* positive p will be globally univalent if the determinant of factor shares $\theta(w)=(w_i a_{ij}(w)/p_j)$ is uniformly bounded away from zero. Both results impose restrictions notably more stringent than in the $n=2$ context largely when some factor prices are at or near zero. Whether this should be a source of comfort is a matter of opinion. Nevertheless, the fact that the higher dimensional case does impose additional restrictions, together with both the pronounced lack of economic intuition in those conditions and also the arcane quality of the analysis used to establish them, has generated a view of global univalence as a quite fragile property.

6.2. Stolper–Samuelson and Rybczynski

To study the comparative statics of equilibria with all n goods produced, differentiate (6.1) noting the cost minimization condition $w(\mathrm{d}A(w)) = 0$:

$$\mathrm{d}p = (\mathrm{d}w)A(w).$$

This can also be written as:

$$\hat{p} = \hat{w}\theta(w), \tag{6.2}$$

where $\hat{p} = (\hat{P}_1, \ldots, \hat{P}_n)$ and $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_n)$. The matrix θ is column stochastic: non-negative with the elements in each column summing to unity. Each column shows the distributive shares of the various factors in the respective industry. If θ is non-singular, (6.2) can be inverted to show the proportional response of nominal factor rewards to proportional price changes:

$$\hat{w} = \hat{p}\theta^{-1}(w). \tag{6.3}$$

Since θ is column stochastic, $u\theta = u$, where u is the vector of ones. But this implies $u = u\theta^{-1}$, i.e. θ^{-1} has unit column sums.

In like manner, the condition for factor market equilibrium, $A(w)X = V$, may be differentiated to obtain, at constant prices and techniques,

$$A(w)(\mathrm{d}X) = \mathrm{d}V. \tag{6.4}$$

This in turn can be written as:

$$\lambda(w)\hat{X} = \hat{V},$$

where $\hat{X} = (\hat{X}_1, \ldots, \hat{X}_n)$ and $\hat{V} = (\hat{V}_1, \ldots, \hat{V}_n)$. The matrix $\lambda = (\lambda_{ji})$ is row stochastic, with each row showing the fraction of aggregate demand for the respective factor contributed by the various sectors. If λ is nonsingular, (6.4) can be inverted to show the proportional effects on commodity outputs of proportional changes in factor endowments:

$$\hat{X} = \lambda^{-1}(w)\hat{V}. \tag{6.5}$$

The row stochastic nature of λ implies that λ^{-1} has unit row sums.

For any vector d, let $I(d)$ denote the square matrix with the elements of d along its diagonal and zeroes elsewhere. Then

$$\theta \equiv I(w)A(w)I(1/p),$$

where $1/p = (1/P_1,\ldots,1/P_n)$. Similarly $\lambda = I(1/V)A(w)I(X)$. With w, p, X and V all strictly positive, the matrices A, θ and λ are all singular or non-singular together. Also, in the latter case, $\theta^{-1} = I(p)A^{-1}I(1/w)$ and $\lambda^{-1} = I(1/X)A^{-1}I(V)$ so that A^{-1}, θ^{-1} and λ^{-1} all have the same sign pattern. These conclusions furnish the basis for a possible duality between $n \times n$ generalizations of the Stolper–Samuelson and Rybczynski theorems.

Now consider how the Stolper–Samuelson theorem might be reformulated for the $n \times n$ environment. When $n = 2$ any change in relative commodity prices will increase one factor reward in terms of both goods and reduce the other factor reward in terms of both goods. To extend fully this result to the higher dimensional context we would wish any change in relative commodity prices to cause each factor price to either rise or fall in terms of all goods. That is, we want θ^{-1} to have the property that, for *every* vector $\hat{p}$ not proportional to u, *no* component of $\hat{w}$ in (6.3) is a weighted average of the components of $\hat{p}$. When $n = 2$ it is also true that each real factor reward changes in the same direction as a particular relative commodity price – that of the good which makes relatively intensive use of the factor. To preserve this aspect one might require a one-to-one correspondence of goods and factors such that the ranking of the components of $\hat{p}$ always coincides with the ranking of the corresponding components of $\hat{w}$.

This is the full-strength higher dimensional version of the 2×2 Stolper–Samuelson property. Obviously a full-strength Rybczynski analog may be stated as well. Once these natural versions have been formulated, one obtains a striking result: they cannot in fact hold. That is, for any technology with more than two goods, there is always *some* change in relative commodity prices which will increase the reward of *some* factor in terms of one good while lowering it in terms of another good. To see this, write out the first equation of (6.3):

$$\hat{w}_1 = \theta^{11}\hat{P}_1 + \cdots + \theta^{n1}\hat{P}_n, \tag{6.6}$$

where θ^{ij} denote elements of θ^{-1}. Suppose $\theta^{11} \neq 0$. Then set $\hat{P}_1 = (1 - a\theta^{21})/a\theta^{11}$, $\hat{P}_2 = 1$, $\hat{P}_3 = \cdots = \hat{P}_n = 0$, where a is any number exceeding unity. Then from (6.6):

$$\hat{w}_1 = \theta^{11}\hat{P}_1 + \theta^{21} = \left[(1 - a\theta^{21})/a\right] + \theta^{21} = 1/a.$$

Thus $\hat{P}_2 > \hat{w}_1 > \hat{P}_3 = \cdots = \hat{P}_n$, i.e. w_1 has fallen in terms of good 2 and risen in terms of good 3. Note that the argument depends crucially upon the existence of at least three goods.

The Stolper–Samuelson result in its strongest form is inherently a 2×2 property and offers no hope for generalization. Thus even "strong" generalizations must sacrifice something relative to the two dimensional environment in order to be guaranteed by sufficient conditions that are not vacuous. Following the seminal work of Chipman (1964, 1969) a number of alternative strong generalizations have been investigated in detail. The one that has received the most attention reduces mathematically to the stipulation that, for some numbering of goods and factors, the matrix θ^{-1} possess positive diagonal elements and negative off-diagonal elements, that is, be what is called a Minkowski matrix.

If θ^{-1} is Minkowski, then the positive diagonal elements will in fact exceed unity, by virtue of the unit column sums. Thus an increase in any commodity price, with all other commodity prices constant, will raise the reward of the corresponding factor in terms of all goods and will lower the reward of every other factor in terms of all goods. This is why the Minkowski property has interested Chipman and his followers. Our earlier discussion establishes that equally clear-cut conclusions do not apply to arbitrary changes in relative commodity prices. Exact economic characterizations have, quite surprisingly, not been featured in the literature, so it is necessary to provide some detail. Start with the following exact characterization.

Proposition 9

The matrix θ^{-1} is Minkowski if and only if, for *any* division of the n goods into two groups, a uniform proportional increase in the prices of all goods in one group relative to all goods in the second group causes the rewards of all factors corresponding to the first group to increase in terms of all goods and the rewards of all factors in the second group to fall in terms of all goods.

To prove this, suppose first that θ^{-1} is in fact Minkowski and let J denote an arbitrary non-empty proper subset of $\{1,\ldots,n\}$. Two dimensional price changes can be represented in full generality by vectors $\hat{p}$ such that $\hat{P}_i = a$ if $i \in J$ and $\hat{P}_i = 0$ if $i \notin J$ for positive a. From (6.3) we have $\hat{w}_j = \sum_{i=1}^{n} \hat{P}_i \theta^{ij} = a\sum_{i\in J}\theta^{ij}$. Since θ^{-1} is Minkowski, and has unit column sums, $\sum_{i\in J}\theta^{ij} > 1$ if $j \in J$ and $\sum_{i\in J}\theta^{ij} < 0$ if $j \notin J$. Thus $\hat{w}_j$ either exceeds each component of $\hat{p}$ or falls short of each component, according as j is in J or not.

Suppose next that the stipulated property regarding price changes holds. Then it must hold in particular for $\hat{p}$ such that $\hat{P}_i = 1$ for some i and $\hat{P}_j = 0$ otherwise. Thus $\theta^{ii} > 1$ and $\theta^{ji} < 0$ for $j \neq i$. Repeating the argument for each i establishes that θ^{-1} must be Minkowski.

A dual Rybczynski characterization follows in analogous fashion.

Proposition 10

The matrix λ^{-1} is Minkowski if and only if, for *any* division of the two factors into two groups, a uniform proportional increase in the endowments of all factors in one group relative to all factors in the second group causes the outputs of all goods corresponding to the first group to rise relative to all factor endowments and the outputs of the other goods to fall relative to all endowments.

Proposition 9 reveals that formulating the generalized Stolper–Samuelson theorem as the specification of conditions for which θ^{-1} is a Minkowski matrix is essentially a hunt for the circumstances under which the strong conclusions about factor rewards apply to the class of two-dimensional relative commodity price changes. The analogous Rybczynski problem likewise looks at the consequences of two-dimensional endowment changes. To approach the problem in this way is in a sense to admit defeat from the start, since a basic purpose in moving to higher dimensions in the first place is to pose questions which do not arise in two dimensions.

Nevertheless this formation does have several advantages. First, nontrivial circumstances do exist when the strong results on factor rewards apply to the class of two dimensional price changes, and we have already seen that they cannot apply to the class of all relative commodity price changes. There has as yet been no attempt to find conditions under which the strong results on factor rewards apply to a class of price changes less than universal but more extensive than the two dimensional one.

Secondly, this formulation is mathematically convenient because it amounts only to looking for a particular sign pattern for θ^{-1}. Furthermore that sign pattern has applications to other problems in economics.

Third, the present formulation possesses a strong duality. The matrices θ^{-1}, A^{-1}, and λ^{-1} all share the same sign pattern and furthermore the latter has unit row sums. Thus, if λ^{-1} has all positive diagonal elements and negative off diagonal elements, the former in fact exceed unity. The strong results on factor rewards necessarily follow from all two-dimensional relative commodity price changes if and only if analogous strong results on commodity outputs necessarily follow from all two-dimensional changes in factor endowments. Furthermore, the same association of factors and goods applies in both cases.

Finally, the present formulation can be given an alternative exact economic characterization. It is impossible to preserve in higher dimensions the full-strength results on factor rewards with respect to all relative commodity price changes, so I posed a problem that requires the former with respect to only a subset of the latter. An alternative strategy would be a formulation that accepted weakened conclusions about factor rewards in exchange for an applicability to all relative commodity price changes. Duality helps show that such an alternative formulation can also be expressed mathematically as the requirement that θ^{-1} be Minkowski, and is therefore completely equivalent to the first formulation.

Proposition 11

The matrix θ^{-1} is Minkowski if and only if, for *any* change in relative commodity prices, whenever the goods are divided into two groups with all those in the first group strictly increasing in price relative to all those in the second group, the total absolute income of the factors associated with the first group is more than sufficient to purchase whatever assortment of goods it was originally used to purchase, and the total absolute income of the factors associated with the second group can no longer buy what it initially did.

Then the first group of factor rewards *on average* increase in real terms and the second group on average fall, but nothing is claimed about individual factor rewards.

To prove the proposition, note that $\mathrm{d}P = \mathrm{d}wA$ or $\mathrm{d}w = \mathrm{d}PA^{-1}$ from which

$$(\mathrm{d}w)I(V) = (\mathrm{d}P)I(X)I(1/X)A^{-1}I(V) = (\mathrm{d}P)I(X)\lambda^{-1}.$$

Then for any non-empty proper subset J of $\{1,\dots,n\}$,

$$\sum_{j\in J}(\mathrm{d}w_j)V_j = \sum_{j\in J}\sum_{i=1}^{n}(\mathrm{d}P_i)X_i\lambda^{ij} = \sum_{i\in J}(\mathrm{d}P_i)X_i\Big(\sum_{j\in J}\lambda^{ij}\Big) + \sum_{k\notin J}(\mathrm{d}P_k)X_k\Big(\sum_{j\in J}\lambda^{kj}\Big) \tag{6.7}$$

and

$$\sum_{j\notin J}(\mathrm{d}w_j)V_j = \sum_{i\in J}(\mathrm{d}P_i)X_i\Big(\sum_{j\notin J}\lambda^{ij}\Big) + \sum_{k\notin J}(\mathrm{d}P_k)X_k\Big(\sum_{j\notin J}\lambda^{kj}\Big). \tag{6.8}$$

Now suppose that θ^{-1} is Minkowski, and let $\mathrm{d}p$ be any change in relative commodity prices and J any non-empty proper subset of goods such that $\mathrm{d}P_j/P_j > \mathrm{d}P_i/P_i$ if $j \in J$ and $i \notin J$. We can normalize prices so that $\mathrm{d}P_j > 0$ when $j \in J$ and $\mathrm{d}P_j < 0$ otherwise. Now λ^{-1} has unit row sums and possesses the Minkowski sign pattern since θ^{-1} does. Thus $\sum_{j\in J}\lambda^{ij}$ exceeds unity if $i \in J$ and is negative if $i \notin J$, and the reverse is true of $\sum_{j\notin J}\lambda^{ij}$. Then from (6.7):

$$\sum_{j\in J}(\mathrm{d}w_j)V_j > \sum_{i\in J}(\mathrm{d}P_i)X_i > 0,$$

and from (6.8):

$$\sum_{j\notin J}(\mathrm{d}w_j)V_j < \sum_{k\notin J}(\mathrm{d}P_k)X_k < 0.$$

Next suppose that the stated property holds and, for arbitrary i, consider the price change $\mathrm{d}P_i > 0$ and $\mathrm{d}P_j = 0$ for $j \neq i$. Then from (6.7) $\mathrm{d}w_i V_i = \mathrm{d}P_i X_i\lambda^{ii}$ which requires $\lambda^{ii} > 1$ for the property to hold irrespective of initial conditions. Repeating the argument for each i establishes that all diagonal elements of λ^{-1}

exceed unity. Next, again for arbitrary i, consider the price change $\mathrm{d}P_i = 0$ and $\mathrm{d}P_k < 0$ for $k = i$. Then from (6.7) $\mathrm{d}w_i V_i = \sum_{k \neq i}(\mathrm{d}P_k)X_k\lambda^{ki}$. If any λ^{ki} were positive, $\mathrm{d}w_i V_i$ would be made negative by appropriate choice of the $\mathrm{d}P_k$, so that the assumed property would not hold for all initial conditions. Repeating the argument for each i establishes that the off-diagonal elements of λ^{-1} are non-positive. Then λ^{-1} has the Minkowski sign pattern and so θ^{-1} does as well.

In analogous fashion a dual Rybczynski result may be obtained. That is, from $\mathrm{d}V = A\,\mathrm{d}X$ it follows that:

$$I(P)\,\mathrm{d}X = I(P)A^{-1}I(1/w)I(w)\,\mathrm{d}V = \theta^{-1}I(w)\,\mathrm{d}V,$$

from which, for any subset J of $\{1,\dots,n\}$,

$$\sum_{i \in J} P_i\,\mathrm{d}X_i = \sum_{j \in J}\left(\sum_{i \in J}\theta^{ij}\right)w_j(\mathrm{d}V_j) + \sum_{k \notin J}\left(\sum_{i \in J}\theta^{ik}\right)w_k(\mathrm{d}V_k) \tag{6.9}$$

and

$$\sum_{i \notin J} P_i\,\mathrm{d}X_i = \sum_{j \in J}\left(\sum_{i \notin J}\theta^{ij}\right)w_j(\mathrm{d}V_j) + \sum_{k \notin J}\left(\sum_{i \notin J}\theta^{ik}\right)w_k(\mathrm{d}V_k). \tag{6.10}$$

These expressions yield the following proposition.[15]

Proposition 12

The matrix λ^{-1} is Minkowski if and only if, for any change in factor endowments where the endowments in one group of factors all increase and the remaining endowments do not increase, at constant prices, the total value of the output of all those goods associated with the first group of factors increases by a larger absolute amount than does national income, and the total value of the remaining goods rises less than does national income.

Thus if all goods are normal, the total excess demand for the first set of goods falls and that of the second rises.

Much work has been devoted to the investigation of technological properties associated with θ^{-1} having the desired sign pattern. One result is immediate from the first economic characterization. Let J be an arbitrary proper subset of $\{1,\dots,n\}$ and let $\hat{p}$ be such that $\hat{P}_j = \varepsilon$ for $j \in J$ and $\hat{P}_j = -\varepsilon$ for $j \notin J$, where $\varepsilon > 0$. Since $\hat{p}$ constitutes a two-dimensional price change, the desired sign pattern for θ^{-1} implies that $\hat{w}_j > \varepsilon$ for $j \in J$ and $\hat{w}_j < -\varepsilon$ for $j \notin J$, where $\hat{w} = \hat{p}\theta^{-1}$. Thus

[15]For convenience the proposition is stated with respect to changes where some endowments increase and some fall, but it can be reformulated to apply to more general cases.

$\hat{p} = \hat{w}\theta$ gives:

$$\sum_{j \in J} w_j \theta^{ji} > \sum_{j \notin J} (-\hat{w}_j)\theta^{ji}, \qquad \text{if } i \in J,$$

$$\sum_{j \notin J} (-\hat{w}_j)\theta^{ji} > \sum_{j \notin J} \hat{w}_j \theta^{ji}, \qquad \text{if } i \notin J.$$

That is, θ necessarily satisfies Uekawa's characterization of a P matrix, discussed in Section 6.1. Thus if θ^{-1} is Minkowski, θ has all positive principal minors (and therefore A and λ do as well). If this holds for all factor prices the technology must be globally univalent, by the Gale–Nikaido theorem. That the strong conclusions about factor rewards always apply to all two-dimensional price changes is then a stronger property than global univalence.

An exact characterization is found in Uekawa, Kemp and Wegge (1973). The matrix A will have an inverse with positive diagonal elements and negative off-diagonal elements if and only if for *any* non-empty proper subset J of $\{1, \ldots, n\}$ and for any positive numbers $\bar{x}_j$, where $j \in J$, there exist positive numbers x_j, where $j \notin J$, such that:

$$\sum_{j \in J} a_{ij}\bar{x}_j > \sum_{j \notin J} a_{ij}x_j, \quad \text{if } i \in J,$$

$$\sum_{j \in J} a_{ij}\bar{x}_j < \sum_{j \notin J} a_{ij}x_j, \quad \text{if } i \notin J.$$

That this condition is strictly stronger than the characterization of a P matrix is evident in that the former must hold for all choices of positive $\bar{x}_j$ whereas the latter only requires some values that work.

The attempt to extend to higher dimensions the strong Stolper–Samuelson property that commodity price changes produce unambiguous changes in all factor rewards thus runs into serious limitations. First, either the class of applicable relative commodity price changes was restricted to the two-dimensional one or the conclusions about factor rewards were obtained only on average; secondly, the presence of the strong property for this restricted class was shown to be equivalent to the imposition on the technology of a strong factor intensity condition that can be interpreted as requiring the technology to be in some sense essentially two dimensional. The most significant accomplishment of this branch of international trade theory must surely be the basic elucidation of the notion that the strong Stolper–Samuelson property is in its very essence largely a two dimensional one.

This accomplishment is basically a destructive one: the central motivation advanced in this chapter for moving to higher dimensions is the widespread view that the salient facts of actual trade cannot be forced into the 2×2 model, so that the 2×2 questions are not really the relevant ones. Still one is sometimes interested in what basically are 2×2 questions. To this extent it is natural to buy more generality by restricting the range of relevant questions even more than in Propositions 9–12. Thus one can obtain definite results without strong conditions on technology if one is prepared *both* to admit only two dimensional price changes (or two-dimensional endowment changes) and *also* to look at conclusions about factor rewards only on average (or only at the output changes in a pair of Hicksian composites).[16] Alternative, basically "strong", extensions of the Stolper–Samuelson result have also been considered.[17] One natural possibility is to reverse the pattern of signs: require θ^{-1} to possess negative diagonal elements with all off-diagonal elements greater than unity. This is equivalent to requiring some association of goods and factors such that any two-dimensional relative price change reduces, in terms of all goods, the rewards of each factor associated with a good whose price has risen and increases, in terms of all goods, the reward of each factor associated with a good whose relative price has declined. The alternative equivalent formulation is in like manner reversed. When $n = 2$ this property is essentially the same as the earlier, differing only in the numbering of goods and factors. But when $n > 2$ the two properties are distinct, and indeed mutually exclusive, because the number of off-diagonal elements exceeds the number of diagonal elements.

Despite the obvious symmetry, this alternative is in some ways less attractive than the earlier formulation. For one thing, the fact that θ^{-1} is column-stochastic does not allow one to infer from its sign pattern that the positive off-diagonal elements in fact exceed unity. This must be imposed as a separate requirement. Thus this alternative formulation does not also reduce mathematically to simply requiring that θ^{-1} have a certain sign pattern. Secondly, the similarity of sign patterns between θ^{-1} and λ^{-1} does not imply that if the positive elements of the former exceed unity those of the latter do as well. Thus the strong duality of the earlier formulation is not retained, and Propositions 11 and 12, which exploited this duality, must be suitably rephrased. As it has happened, however, the literature has in fact strangely been concerned only with the mathematically more convenient problem of assuring that θ^{-1} have negative diagonal and positive off-diagonal elements. That the latter also exceed unity, central to the economic motivation, has not thus far even been investigated.

Further formulations apply to even more restrictive classes of price changes than the two dimensional one. One alternative is to require that θ^{-1} have no elements between zero and unity: any change in the price of a *single* commodity relative to all others should cause each factor reward to alter unambiguously in real terms. Still another alternative requires only that all diagonal elements exceed

[16] Details may be found in Neary (1979), which has pioneered this interesting approach.
[17] See Chipman (1969), Uekawa (1971, 1979), Inada (1971), Ethier (1974), and Egawa (1978).

unity (or yet again that they each exceed both unity and every off-diagonal element in the same row). That is, any change in the price of a single good relative to all others should increase in terms of all goods the reward of the associated factor (and increase every other factor reward a smaller amount).

6.3. *The Heckscher–Ohlin theorems*

In two dimensions the Heckscher–Ohlin theorems impose more restrictive assumptions than do the other fundamental propositions, so it should come as no surprise, in view of the difficulties already seen, that very little has been attempted in the way of strong generalizations of the former to higher dimensions. But what can be gleaned from the comparative statics results that have been obtained?

Suppose that the technology is indeed such that θ^{-1} always exists and has the Minkowski sign pattern for some numbering of goods and factors. Consider first the price version of the Heckscher–Ohlin theorem. In two dimensions this proposition can be thought of as consisting of two parts: the contention that differences between countries in autarkic relative factor prices correspond to differences in autarkic relative commodity prices, and the assertion that the latter predict trade patterns. The second part – a portion of what is often called the Law of Comparative Advantage – is applicable to more general circumstances than factor-endowment models. Its higher-dimensional fate was discussed earlier in this chapter. Focus now on the first part: the link between autarkic commodity price differences between countries and relative factor abundance as measured by autarkic factor prices. When the former are of the appropriate two-dimensional sort (and small) we can call on Proposition 9.

Proposition 13

Suppose that autarkic commodity prices differ in a two-dimensional fashion between two countries with an identical technology for which θ^{-1} exists and is Minkowski for some numbering of goods and factors. Then in each country the autarkic reward of each factor associated with each of the goods more costly in that country is greater, in terms of each good, than the reward of each other factor.

Next, the quantity version. Since global univalence is implied if λ^{-1} is required to have the Minkowski sign pattern, two countries will have equal factor prices if they both produce all n goods in free trade. Also they will consume the goods in identical proportions if they share identical homothetic preferences. Then Proposition 10 gives a quantity version.

Proposition 14

Suppose that two countries with identical homothetic preferences share a common technology for which λ^{-1} exists and is always Minkowski for some numbering of goods and factors. Suppose that both countries produce all goods in free trade, and that their respective factor endowments differ from each other in a two-dimensional fashion. Then each country exports each good associated with each of that country's relatively abundant factors.

These two results are patently unsatisfactory because they apply only when countries differ in very special ways, and the basic reason for moving to higher dimensions is to study the consequences of more complex differences. Propositions 11 and 12 allow us to examine more general cases, but only at the sacrifice of all attempts to specify the pattern of trade on a commodity-by-commodity basis. For example if several countries engage in free trade with factor price equalization, and the countries have identical homothetic demands, then the bundle of goods consumed by any country will have required for its manufacture a fraction of the world endowment of each factor just equal to that fraction of world income contributed by the country in question. Thus if, for some country, we let $\mathrm{d}V_j = V_j - \gamma V_j^*$ in (6.9) and (6.10), where γ denotes the fraction of world (free trade) income contributed by the country, and V_j and V_j^* denote respectively the national and world endowments of factor j, then the $\mathrm{d}X_i$ will equal net exports. Let J denote the country's relatively abundant factors, i.e. those for which $\mathrm{d}V_j$ as thus defined are positive. Then Proposition 11 gives the following quantity version.[18]

Proposition 15

Suppose that there is free trade between all countries, factor prices are equalized, and λ^{-1} exists and has the Minkowski sign pattern. Then each country is on balance an aggregate net exporter, at world prices, of the set of goods corresponding to those factors which are relatively abundant in that country.

7. General results

The most significant message to come from the previous section is that of a basic failure to break the chains of "twoness" when it is demanded that fairly strong properties survive. This failure was manifest in both the economic meaning of the mathematical properties under consideration and also in the technological restrictions implying those properties. Such a message has seemed devastating to modern trade theory, cast in a 2×2 mold widely regarded as incapable of adequately describing reality. Thus a change in strategy is certainly of potential interest. Instead of attempting to impose results that retain as much of the strength of two dimensions as possible, enquire into what results can be obtained generally under restrictions no more severe than those conventionally adopted in the two dimensional environment.

[18]See McKenzie (1966, pp. 100, 101) and Kemp (1976, pp. 45–77).

7.1. Factor-price equalization

A natural place to start is with factor-price equalization, in contrast to the previous section's discussion of global univalence. That is, consider the higher dimensional generalization of Proposition 3 rather than, as before, of Proposition 4.

Suppose for a moment that several countries, under conditions of free trade, are alike in every way, including factor endowments, so that at existing commodity prices all goods can be produced in all countries, and in each:

$$p = wA(w). \tag{7.1}$$

Factor prices are obviously equalized now; under what circumstances will they remain equalized if endowments are allowed to vary across countries? Let K_w be the cone of factor endowments spanned by the columns of $A(w)$ in (7.1); that is, K_w consists of all vectors V satisfying $V = A(w)X$ for some non-negative X. Suppose now that we vary the endowment vectors of the countries, but that we do this so as to keep all endowments in K_w, so as to hold constant each country's national income, wV, evaluated at the original factor prices, and so as to leave world factor supplies fixed. Such variations generate potential equilibria across which world output, national demands and factor prices do not vary, so that factor price equalization holds. But need these potential equilibria actually obtain? When $n = 2$, any country with an endowment in K_w, and freely trading at the specified commodity prices, necessarily has factor prices equal to w; this is the essential constituent of the factor-price equalization theorem. The property carries over to higher dimensions. To see this, suppose that for some $V \in K_w$ another factor price vector, w', is also consistent with equilibrium, that is, $p = w'A(w')$ and for some non-negative X', $V = A(w')X'$. Since $A(w')$ is the (unique) least-cost set of techniques at factor prices w',

$$w'A(w) \geq p = wA(w), \tag{7.2}$$

with at least one strict inequality if $w' \neq w$. Similarly the fact that $A(w)$ is the least-cost set of techniques at factor prices w yields:

$$wA(w') \geq p = w'A(w'), \tag{7.3}$$

with at least one inequality if $w' \neq w$. Multiplying both sides of (7.2) by X gives $w'V \geq wV$ with strict inequality if $w' \neq w$, and multiplying both sides of (7.2) by X' gives $wV \geq w'V$ with strict inequality if $w \neq w'$. Thus $w = w'$.

For any p, all countries with endowments in an associated K_w [that is, one for which $p = wA(w)$] will have equal factor prices in free trade. This is a substantive result when K_w has full dimension, m. This is so when $A(w)$ has at least m linearly independent columns: the same condition as in the two-dimensional case. [Note that consistently with Section 5, factor-price equalization becomes a fluke when factors outnumber goods, as $A(w)$ necessarily has fewer than m columns.] Proposition 3 carries over to higher dimensions without difficulty as long as there are at least as many goods as factors.[19]

7.2. *Stolper – Samuelson*

7.2.1. *The role of factor intensities*

The conventional Stolper–Samuelson result can be decomposed into two parts: a prediction, based on factor intensities, of the direction of response of relative factor price changes to relative commodity price changes, and the assertion that factor prices move in different directions in real terms. The strong attempts at generalization tended to link these two aspects together: several forms of the relative factor-intensity hypothesis were shown to imply (or be equivalent to) certain strong relations between commodity prices and real factor rewards. In taking a more general approach it proves convenient to keep the two aspects separate. I first ask to what extent is it true that relative factor intensities allow one to infer something about the *direction* of changes in factor rewards from the direction of commodity price changes.[20]

Consider two equilibria characterized by (initial and terminal) goods price vectors p^0 and p^1, and corresponding factor price vectors w^0 and w^1. I impose no restrictions on the relative number of goods and factors, but I do look only at those goods actually produced in both the initial and terminal equilibria (though not necessarily in "intermediate" ones). Thus $p^0 = w^0A(w^0)$ and $p^1 = w^1A(w^1)$. Application of the mean value theorem to the function $b(w) = wA(w)(p^1 - p^0)$ yields the conclusion that, for some factor price vector $\bar{w}$,

$$b(w^1) = b(w^0) + (w^1 - w^0)[A(\bar{w}) + \bar{w}\,\mathrm{d}A(\bar{w})](p^1 - p^0).$$

Now cost-minimization implies that $\bar{w}\,\mathrm{d}A(\bar{w}) = 0$ so that

$$b(w^1) - b(w^0) = (w^1 - w^0)A(\bar{w})(p^1 - p^0).$$

Noting $b(w^1) - b(w^0) = (p^1 - p^0)(p^1 - p^0) > 0$ produces the conclusion:

$$(w^1 - w^0)A(\bar{w})(p^1 - p^0) > 0. \tag{7.4}$$

[19]See McKenzie (1955), Samuelson (1953), Uzawa (1959), Ethier (1974), and Dixit and Norman (1980).
[20]See Ethier (1982).

That is, there is a positive *correlation* between the elements of the vector $(w^1 - w^0)$ and those of $A(\bar{w})(p^1 - p^0)$ – or, equivalently, between $(w^1 - w^0)A(\bar{w})$ and $(p^1 - p^0)$. On average, high values of $w_i^1 - w_i^0$ are associated with high values of both a_{ij} and $P_j^1 - P_j^0$ or with low values of both: there is a tendency for changes in relative commodity prices to be accompanied by increases in the rewards of factors employed most intensively by those goods whose prices have relatively risen the most and employed least intensively by those goods whose relative prices have fallen the most.

Note several aspects of this result. The first is its extreme generality: the correlation is a direct result of cost minimization and requires no special restrictions on either technology or dimensionality. Furthermore the correlation, which becomes a certainty when $n = 2$, is a clear generalization of the 2×2 result. Also, for small enough changes $\bar{w} = w^0$ will work in (7.4): the pattern of factor price changes can on average be predicted from the initial factor intensities. But for large changes this is not so, and it may be necessary to choose a $\bar{w}$ near neither w^1 nor w^0 so that the pattern of intensities might not be observable for some applications. For example, consider the two dimensional case with a factor intensity reversal illustrated in Figure 7.1. A movement from A to B involves the opposite pattern of factor reward changes as one from C to D. In the former case $\bar{w}_1/\bar{w}_2$ must be given by a point such as F whereas in the latter case it would look like E. Finally, note that $p^1 - p^0$ measures the price changes only of those goods actually produced in both states (though not necessarily in intermediate states, such as when $w = \bar{w}$). Thus the relation of real or nominal factor rewards to changes in the prices of other goods, if any, are not described.

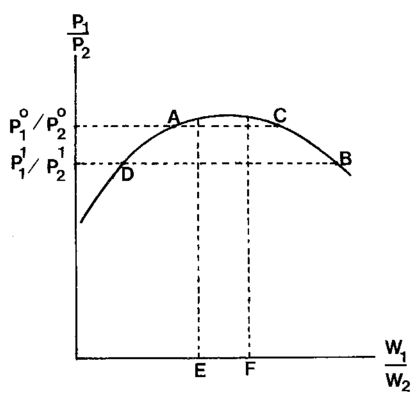

Figure 7.1. The proper choice of w.

7.2.2. *Real rewards*[21]

Consider an increase in the price of some good that is actually produced initially, all other commodity prices remaining unchanged. As the good was produced, price must have initially equalled its cost of production. Cost may now rise above price,[22] causing production to cease, but competition will prevent price rising above cost in any case. Thus the proportional rise in the price of this good, which I number the first, must be no greater than the proportional rise in its cost:

$$\hat{P}_1 \leq \theta_{11}\hat{w}_1 + \cdots + \theta_{m1}\hat{w}_m. \tag{7.5}$$

The θ_{ji} in this relation are non-negative and sum to unity. Thus (7.5) implies that there exists *some* factor – call it the first – such that $\hat{w}_1 \geq \hat{P}_1 > 0 = \hat{P}_2 = \cdots = \hat{P}_n$. Thus it is generally true that a rise in the price of any good initially produced must raise the reward of some factor in terms of every other good and lower it in terms of no good.

To proceed further requires that in the new equilibrium factor 1 be used in the production of some other good, say the second. As this good is produced, its price equals its cost. It may not have been produced initially, so its initial cost could have exceeded its price, but in any case competition prevented its cost from being strictly less than its price. Thus the proportional rise in the latter (zero) must have been at least as great as that in the former:

$$0 = \hat{P}_2 \geq \theta_{12}\hat{w}_1 + \cdots + \theta_{m2}\hat{w}_m. \tag{7.6}$$

We know that $\hat{w}_1 > 0$ and our assumption is that $\theta_{12} > 0$. Since the θ_{j2} are non-negative and sum to unity, (7.6) implies that some factor reward, call it the second, strictly falls: $\hat{w}_2 < 0 = \hat{P}_2 = \cdots = \hat{P}_n < \hat{P}_1$. Thus every good is "friend" to some factor and an "enemy" to some other factor, to use the terminology of Jones and Scheinkman (1977), so that any price rise is conflict-generating.

Proposition 16

A rise in any single commodity price will cause the reward of *some* factor to rise in terms of all other goods and to fall in terms of none, and it will cause the reward of *some* other factor to fall in terms of all goods – provided only that the

[21] References include Ethier (1974), Kemp (1976, chs. 4, 7), Jones and Scheinkman (1977), Diewert and Woodland (1977), Chang (1979), and Jones (1979, chs. 8, 18). For a consideration of joint production, see Jones and Scheinkman (1977), Woodland (1977), and Chang, Ethier and Kemp (1980).

[22] Even this possibility is easily ruled out.

good is initially produced and that every factor which it employs is subsequently also employed elsewhere in the economy.

The distinctive feature of this proposition is its extreme generality: no special condition need be imposed on the technology, and the number of commodities may fall short of, equal, or exceed the number of factors. The proposition is easily extended to encompass all two dimensional price changes as discussed in Section 6. If the prices of some subset J of goods all rise proportionally relative to those of all goods not in J, we clearly require only that some good in J be produced initially and that all factors used to produce this good also be subsequently used to produce some good not in J.

Proposition 16 does leave open the possibility that the favored factor's reward might remain constant in terms of the more expensive good, and, therefore, might not increase in real terms if spent entirely on that good. This remote possibility can be eliminated in either of two ways. One might depart from the Stolper–Samuelson tradition with the mild restriction on demand that all agents always spend increases in incomes on at least two goods. Then a factor reward which falls in terms of no good and which fails to rise in terms of at most one good necessarily increases in real terms. Alternatively one can employ a weak assumption of a purely technological nature: no good employs a *specific assortment* of factors, that is, the factors used in positive amounts by any good are precisely the factors used in positive amounts by some other good. With this assumption we can let good 2 use the same collection of factors as does good one in the above derivation of Proposition 16. Then good one uses factor two, since good two does. Thus in (7.5), $\theta_{21} > 0$ and $\hat{w}_2 < 0$. Then (7.5) in fact requires that for some factor, say the first, $\hat{w}_1 > \hat{P}_1$. It is interesting to note that both methods of resolving the problem involve imposing "twoness" at a *minimum*: income must be spent on *at least two* goods, or an assortment of factors must be common to *at least two* industries.

There remains the crucial problem of determining what can be said about the consequences of relative price changes in general, as opposed to the two-dimensional changes thus far considered. For an arbitrary relative price change, the role of good one in the above discussion can clearly be played by a commodity whose price rises at least as much, proportionally, as any other commodity price, provided that there is such a good initially produced. Similarly the role of good two can be played by any good whose price rises relative to no other, provided that there is such a commodity produced after the price change. Thus our conclusions apply to arbitrary relative price changes as long as the latter are what might be called "non-specializing": some initially-produced good falls in price relative to no other and some terminally-produced good rises in price relative to no other. These various conclusions are summarized in the following.

Proposition 17

Any non-specializing relative price change necessarily raises some factor reward in terms of all goods and lowers some other factor reward in terms of all goods, provided only that no good employs a specific assortment of factors.

Note that the conditions imposed by this proposition are trivially satisfied if all goods use all factors and if all goods are produced (the purpose of the conditions is simply to accommodate "zeros"), as in the usual two dimensional analysis. Thus the propositions meet the goal of employing restrictions no more severe than those used when $n = 2$.

Still more conclusions can be obtained by exploiting the duality embodied in the reciprocity relations derived in Section 1. But before doing this I must turn to the Rybczynski analogs of the present results.

7.3. Rybczynski

The remainder of this section is basically a collection of repetitions and applications of the logic thus far. But the process requires some care because of an asymmetry between the Stolper–Samuelson and Rybczynski problems in one respect. This involves dimensionality. Suppose that the economy is initially in equilibrium at factor prices w. As we have seen, any change in V that leaves the economy in K_w at unchanged commodity prices will also leave factor prices – and thus techniques of production – unchanged as well. Now if K_w has dimension m and if V is initially in its interior, any sufficiently small $\mathrm{d}V$ will leave it there. But when there are fewer produced goods than factors a change in V will, except for a fluke, involve a departure from K_w and therefore require a change in w to maintain factor market equilibrium (the previous Stolper–Samuelson discussion was not sensitive to the relative numbers of goods and factors). This is illustrated by a consideration of the role played by factor intensities in predicting the direction of output changes resulting, at constant commodity prices, from endowment changes.

7.3.1. The direction of output changes

Suppose that $V^0 = A(w)X^0$ and $V^1 = A(w)X^1$ in the initial and terminal states. Here V^0 and V^1 are assumed to both lie in K_w so that factor prices are not altered by the change. Then $V^1 - V^0 = A(w)(X^1 - X^0)$ so that multiplication of both sides by $V^1 - V^0$ gives:

$$(V^1 - V^0)A(w)(X^1 - X^0) > 0. \tag{7.7}$$

Thus we obtain a Rybczynski correlation analogous to the earlier Stolper–Samuelson one: endowment changes tend on average to increase the most those goods making relatively intensive use of those factors which have increased the most in supply, etc. But factor price equalization played a key role in this analysis which is therefore of practical significance only when $n \geq m$.

7.3.2. The magnitude of output changes

Consider an increase in the endowment of some factor, call it the first, that is fully employed after the endowment change. It may or may not have been fully employed before, but demand for the factor can have increased proportionally no less than its supply. Thus, if factor prices are unchanged,

$$\hat{V}_1 \leq \lambda_{11}\hat{X}_1 + \cdots + \lambda_{1n}\hat{X}_n. \tag{7.8}$$

The λ_{1i} are non-negative numbers summing to unity; then (7.8) requires that for some good, call it the first, $\hat{X}_1 \geq \hat{V}_1 > 0 = \hat{V}_2 = \cdots = \hat{V}_m$. So it is generally true that an increase in the endowment of a subsequently fully employed factor must increase the output of some good relatively to every other factor and lower it in terms of no factor, provided that factor prices do not change. If good one also uses an additional factor, say the second, fully employed initially,

$$0 = \hat{V}_2 \geq \lambda_{21}\hat{X}_1 + \cdots + \lambda_{2n}\hat{X}_n. \tag{7.9}$$

Since $\hat{X}_1 > 0$ and by assumption $\lambda_{21} > 0$, (7.9) requires that for some good, say the second, $\hat{X}_2 < 0$.

Proposition 18

At constant factor prices, an increase in any factor endowment will cause the output of *some* good to rise relative to all other factors and to fall relative to none, and will cause the output of *some* other good to fall absolutely – provided only that the factor is subsequently fully employed and that every industry which uses it also uses another factor that is initially fully employed.

This proposition is also extremely general, although its requirement of constant factor prices potentially gives a role to one aspect of dimensionality – the relative number of goods and factors – that did not figure at all in Proposition 16. The argument can be extended in obvious fashion to encompass arbitrary two-dimensional endowment changes.

That the output of the favored commodity actually rise relative to *all* factors, including the expanding one(s), can be guaranteed by an additional assumption: each factor is *non-specific* in the sense that it is used in positive amounts by exactly the same sectors which use some other factor. For if factor two plays this role in the above demonstration, $\lambda_{22} > 0$ implies $\lambda_{12} > 0$ so that $\hat{X}_2 < 0$ requires $\hat{X}_1 > \hat{V}_1$. Finally, the present analysis can encompass arbitrary relative endowment changes in a way analogous to the earlier inclusion of arbitrary relative price changes: the former must be "employment-maintaining" in the sense that some factor fully employed after the change increases proportionally at least as much as every other factor, and some factor initially fully employed increases proportionally no more than any other.

Proposition 19

At constant factor prices, any employment-maintaining change in relative factor endowments necessarily causes some output to expand relative to all endowments and some output to fall relative to all endowments, provided only that all factors are non-specific.

7.4. *General results*

7.4.1. *Even technologies*

Further results depend upon the number of factors and produced goods. Start with the case $n = m$. Factor prices will not change in response to (small) endowment changes if commodity prices are held constant, so Propositions 18 and 19 are now applicable to the latter case. These "Rybczynski" results can be used to obtain additional "Stolper–Samuelson" results, and vice versa, by exploiting the reciprocity relations derived in Section 1:

$$\frac{\partial X_i}{\partial V_j} = \frac{\partial w_j}{\partial P_i} \tag{7.10}$$

(with a common value of a^{ji}). Suppose that all factors are nonspecific. Then for any factor j there exists some commodity i such that $\partial X_i/\partial V_j < 0$, by Proposition 18. Then (7.10) implies that $\partial w_j/\partial P_i < 0$: for any factor there is some commodity an increase in whose price will lower the real reward of that factor. Each factor has an "enemy", in the terminology of Jones and Scheinkman.[23] Thus it is possible to control the real income of any factor by varying a single commodity price in the opposite direction.

Proposition 18 also establishes that for any factor j there is a good i such that $\partial X_i/\partial V_j > X_i/V_j > 0$. From this (7.10) implies that $\partial w_j/\partial P_i > X_i/V_j > 0$, but this does not assure that w_j rises relative to P_i. Accordingly we do not have

[23]This result also follows directly from $\theta\theta^{-1} = I$ without recourse to the reciprocity relations.

another unambiguous statement about real factor rewards, but a quite strong conclusion still follows. For, from (7.10)

$$\frac{\partial w_j}{\partial P_i}V_j - X_i = \left(\frac{\partial X_i}{\partial V_j}\frac{V_j}{X_i} - 1\right)X_i,$$

so that $\partial X_i/\partial V_j > X_i/V_j$ implies that $V_j \partial w_j/\partial P_i > X_i$. Now the national product function $y(p, V)$ has the property $X_i = \partial y/\partial P_i = \sum_{k=1}^{n} V_k \partial w_k/\partial P_i$. Thus $\sum_{k \neq j} V_k \partial w_k/\partial P_i < 0$: the increase in P_i must cause w_j to rise by so much that the aggregate income of all other factors falls. Thus these incomes no longer suffice to purchase the bundle of goods they purchased before, so that, *if* after the rise in P_i the entire country can still afford its original consumption bundle, the real reward of factor j must have risen. But in general the strong positive relation between P_i and w_j does not suffice to rule out the possibility that, if the rise in P_i is associated with a fall in national income and if w_j fails to rise proportionally more than P_i, the expenditure of income accruing to factor i might be sufficiently concentrated upon good j to prevent a rise in the factor's real reward.

The phalanx of Stolper–Samuelson conclusions requires both Propositions 16 and 17's demand that no good employs a specific assortment of factors and Propositions 18 and 19's demand that all factors are non-specific. Thus the total requirement is that $A(w)$ be what can be called *fully-latticed*: a non-zero matrix in which no row or no column has a unique sign pattern.

A similar approach exploits duality to use the Stolper–Samuelson results to extend the Rybczynski results. In the end the following emerges.

Proposition 20

Suppose $n = m$ and that $A(w)$ is fully latticed. Then

(i) to each good there corresponds some factor such that an increase in the good's price raises the factor reward to a greater degree, and an increase in the factor endowment at constant commodity prices raises the output of the good by enough so that the total value of all other outputs falls;

(ii) to each good there corresponds some factor such that an increase in the good's price lowers the factor's reward, and an increase in the factor's endowment at constant commodity prices lowers the output of the good;

(iii) to each factor there corresponds some good such that an increase in the good's price lowers the factor's reward, and an increase in the factor's endowment at constant commodity prices lowers the output of the good;

(iv) to each factor there corresponds some good such that an increase in the good's price raises the factor's reward so much that the aggregate income of all other factors falls, and an increase in the factor's endowment raises the output of the good in greater proportion.

With a fully latticed, and even, technology, each good has a friend and an enemy among the factors in a Stolper–Samuelson sense, and each factor has some enemy and a "qualified" friend. In a Rybczynski sense each factor has a friend and an enemy, and each good has some enemy and a "qualified" friend.

7.4.2. Odd technologies: More goods than factors

Propositions 16 and 17, insensitive to dimensionality, remain valid, if the number of produced goods exceeds the number of factors. At constant commodity prices, endowment changes leave factor prices unaltered; thus Propositions 18 and 19 continue to describe the output effects of changes in factor supplies when the prices of goods are constant. It is true that, as indicated in Section 4, outputs – and therefore output changes as well – are now indeterminate. But this does not affect the validity of the propositions, which assert that *some* outputs must respond in certain fashions, but which do not claim that their identifies can be determined.

It is the body of results linked to duality that now require more care: for example, the left-hand terms in expressions such as (7.10) are now undefined. Proceed as follows.[24] Choose any subset, containing m goods which remain produced, out of the total of n goods. For this subset, of course,

$$(\mathrm{d}p) = (\mathrm{d}w)A(w), \tag{7.11}$$

where $A(w)$ is $m \times m$. Each good, not included in the subset, which also remains produced furnishes another relation between the respective $\mathrm{d}P_i$ and $\mathrm{d}w$, but this is additional information which does not influence (7.11) and which one can choose to ignore. Next consider the exercise of varying factor endowments subject to constant commodity prices (and thus factor prices). For each of the goods that was ignored in writing (7.11) set $\mathrm{d}X_i = 0$. If $\mathrm{d}X$ denotes the m-vector of changes in the other goods,

$$\mathrm{d}V = A(w)\,\mathrm{d}X, \tag{7.12}$$

where $A(w)$ is the same as in (7.11). The $n - m$ excluded goods use an unchanging bundle of factors in their production. Ignoring this bundle gives a standard $m \times m$ submodel for which the reciprocity relations can be derived in the usual way. (Geometrically, the basic $n \times m$ production model is being invested with the requisite strict quasi-convexity by projection into an appropriate subspace; this can obviously also be done by setting the $n - m$ "extra" $\mathrm{d}X_i$ equal to specified values other than zero.) The earlier derivation of results from duality can now be repeated and thereby extended to the $n > m$ case. The extra goods require one change in interpretation: the fact that for each good i there exists, from Proposi-

[24] See also Chang (1979, pp. 718–723).

tion 17, some factor j such that $\partial w_j / \partial P_i < 0$ now implies, from (7.11), that for each good i there is some factor j an increase in whose endowment *allows* a reduction in X_i. The indeterminancy of production patterns precludes a stronger statement.

Proposition 21

Proposition 20 remains fully valid when $n > m$, with the sole modification that, in parts (i) and (ii), an endowment increase at constant commodity prices *allows* the indicated response in outputs, but does not require them.

7.4.3. *Odd technologies: More factors than goods*

The basic even results can accommodate additional produced goods with only the slightest discomfort. The same is not true of additional factors. Suppose that $m > n$. This, again, has no effect on Propositions 16 and 17. Propositions 18 and 19 also remain valid, but they cease to apply to endowment changes with constant commodity prices, because factor prices must now change under such circumstances to maintain factor market equilibria. As Kemp and Wan point out [in Kemp (1976, p. 56)], it is necessary to choose between two mutually exclusive approaches, each faithful to 2×2 practice in one respect and unfaithful in another. One possibility is to arbitrarily hold factor prices fixed when endowments change at constant commodity prices. This requires consideration of factor market disequilibrium; in effect one moves to a "Keynesian" environment from the usual "Walrasian" one.

In this case Propositions 18 and 19 remain relevant. They should be viewed as comparisons of "Keynesian" equilibria with different factor demands – the appropriate interpretation for endowments in this context. What this approach must sacrifice are the results due to duality: the $\partial X_i / \partial V_j$ terms in the reciprocity relations (7.10) are derived on the assumption of constant commodity prices and continuous factor market equilibrium (and therefore changing factor prices if $m > n$). Thus those parts of the Stolper–Samuelson and Rybczynski results that followed from the application of (7.10) to Propositions 16 and 18 do not survive.

The alternative approach is to compare situations involving factor-market equilibria, and therefore altered factor prices. Propositions 18 and 19 now cease to apply, so that portion of the Rybczynski results is lost. But now the reciprocity relations can be applied, so that the portion of the Rybczynski results obtained by duality from Propositions 16 and 17 once again hold.

In sum, when there are more factors than goods about half the comparative statics results survive. On the Stolper–Samuelson side, it remains true that each good is a friend to some factor and an enemy to another, but each factor need no longer have some enemy and some "qualified" friend. On the Rybczynski side, if factor markets are required to clear, each good has an enemy and a "qualified" friend, but every factor need no longer be a friend to some good and an enemy to another; when factor prices are instead held fixed, just the reverse is true.

7.4.4. *The multi-commodity specific-factors model*

The multi-commodity extension of the specific-factors model discussed in Section 5.2 is instructive. Suppose that a positive endowment is available of each of n factors specific to each existing good, and that the technology is such that at all positive commodity price vectors positive amounts of all goods are produced. This structure is simple enough that one readily understands that the properties discussed in Section 5.2 are essentially unchanged.[25]

This model violates both of the conditions of Proposition 20: $n < m$ ($= n+1$) and $A(w)$ is not fully latticed. But the former violation is the crucial one. Each good is a Stolper–Samuelson friend to its own specific factor and an enemy to all other specific factors; the mobile factor has neither friend nor enemy. In a Rybczynski sense, with factor markets required to clear, each good has as a qualified friend its own specific factor and every other specific factor as an enemy, while the mobile factor is neither friend nor enemy to any good. These possibilities arise because $m > n$, with the simple structure of the model assuring that the possibilities materialize in a transparent way.

The requirement that $A(w)$ be fully latticed places weak bounds on the degree of specificity: no good may use a unique collection of factors, but the collection may be shared with as few as one other sector, and so forth.[26] The specific-factors model is a limiting case with specificity so pervasive as to violate even this weak requirement. The interesting thing is how little difference it seems to make. The only purpose of requiring $A(w)$ to be fully latticed, you will recall, was to prevent "ties" between proportional increases in commodity prices (factor endowments) and factor rewards (outputs). In the specific factors model these ties nevertheless do not arise, except for the Rybczynski experiments where factor prices are kept fixed and the market for the mobile factor allowed not to clear. Then a change in any specific factor would tend to induce an equiproportional change in the output of the corresponding good and no changes in other outputs.

7.5. *The Heckscher–Ohlin theorems*

The Heckscher–Ohlin theorems explain the pattern of comparative advantage on the basis of factor endowments. In higher dimensions comparative advantage generalizes as a correlation between price differences and trade flows. It is

[25] For details of this model see Samuelson (1971) or Jones (1975).
[26] For a discussion of comparative statics and specificity see Fishburn and Kemp (1977).

therefore natural to enquire how endowment differences are related, in an average sense, to trade.

7.5.1. The price version

Start with an examination of factor abundance as revealed by autarkic factor prices. Recall (7.4) from the earlier Stolper–Samuelson discussion:

$$(w^1 - w^0)A(\bar{w})(p^1 - p^0) > 0. \tag{7.4}$$

Interpreting w^1 and w^0 as autarkic factor prices in the home and foreign countries, expression (7.4) establishes that countries tend to have a comparative advantage in goods intensive in their use of relatively abundant factors. Thus the price version generalizes to this degree without difficulty.

Comparative advantage is in turn correlated with the pattern of trade. To link directly factor abundance to trade flows use logic like that preceding (7.4) to obtain:

$$(p^1 - p^0)M = (w^1 - w^0)A(\bar{w})M,$$

where w^1 and w^0 denote home and foreign autarky factor prices and $M = D - X$, the vector of home imports. The discussion of comparative advantage in Section 2 established that $(p^1 - p^0)M \geq 0$ [recall expression (2.5)]. Then

$$(w^1 - w^0)A(\bar{w})M \geq 0. \tag{7.13}$$

Thus countries on average tend to import goods that make relatively intensive use of relatively scarce factors. Recall the earlier discussion of (7.4): to avoid problems arising from the presence of higher-dimensional analogs of factor intensity reversals, $\bar{w}$ must be chosen properly.

Expression (7.13) asserts that, for some "intermediate" factor price vector the price version of the Heckscher–Ohlin theorem holds. But no procedure is given for finding $\bar{w}$, that is, for measuring the relevant pattern of factor intensities. An alternative, constructive, path to generalization proceeds as follows.[27] Construct the technology matrix $\bar{A}$ on the basis of the country of origin of traded goods: the ith column of $\bar{A}$ equals the foreign technique of production if good i is imported and the domestic technique otherwise. Thus $\bar{A}$ equals the actual A matrix in each country if trade has equalized factor prices but will be a composite of the two A matrices otherwise. Define the factor content of trade as: $M_V = \bar{A}M$.

Since the two countries share a common technology, it must be feasible to produce D from the factors $V + M_V$, simply by ceasing to produce exports and

[27]The basic reference here is Deardorff (1981). See also Appendix One of Ethier (1983).

producing imports in the same way they are produced abroad. Of course such a process will not generally be profitable at home autarkic prices, w^{A}, p^{A}:

$$w^{\mathrm{A}}(V+M_V) \geq p^{\mathrm{A}}D.$$

Now $p^{\mathrm{A}}D \geq p^{\mathrm{A}}D^{\mathrm{A}}$, where D^{A} denotes the vector of goods consumed in autarky, by the usual gains-from-trade argument. Thus $w^{\mathrm{A}}M_V \geq p^{\mathrm{A}}D^{\mathrm{A}} - w^{\mathrm{A}}V = 0$. The same logic applied to the rest of the world yields $w^{\mathrm{A}^*}M_V^* \geq 0$. Since $M_V = -M_V^*$,

$$(w^{\mathrm{A}} - w^{\mathrm{A}^*})M_V \geq 0. \tag{7.14}$$

This is a second price version, dealing with the factor content of trade: countries tend to export (indirectly via commodities) their relatively abundant (in a price sense) factors. Substitute for M_V in the correlation (7.14) to examine the pattern of commodity trade:

$$(w^{\mathrm{A}} - w^{\mathrm{A}^*})\bar{A}M \geq 0. \tag{7.15}$$

This third price version says that, on average, countries tend to export goods which make relatively intensive use of relatively abundant factors.

Note that (7.14) and (7.15), like (7.13), are extremely general as they require no special assumptions on technology nor any relation between m and n. Also, like the 2×2 price version, nothing need be said about demand. But it is important that factor content be measured according to the *country of origin* of goods. This is done to avoid the problems introduced by higher-dimensional analogs of factor-intensity reversals, which must be assumed away in the 2×2 case. Consider, in the latter context, a separating factor intensity reversal between the two countries. Then (7.14) says that each country exports its relatively abundant factor: impossible for both countries if each calculates the factor content of trade according to its own techniques in use, or if the techniques of one of the countries are used by both. But the prediction is true when, as in (7.14) and (7.15), the technique of the exporting country is used for each good.

7.5.2. *The quantity version*

Even in the 2×2 environment the quantity version normally is based on an assumption of identical homothetic demands across countries, so do the same now. That is, tastes in both countries can be represented by a common set of radially symmetric indifference surfaces.[28]

[28] See Dixit and Norman (1980, pp. 96–100) and Woodland (1982, ch. 7) for treatments of the quantity version.

Recall that the national product function $y(p, V)$ measures the minimum that must be paid to factors V over all factor rewards that leave costs no lower than p. Thus $w^{A*}V \geq y(p^{A*}, V)$ where p^{A*} denotes foreign autarkic commodity prices. Next, if u_0 solves $y(p^{A*}, V) = e(p^{A*}, u_0)$ it follows that home autarkic utility $u_A \leq u_0$ since the opportunity to transact at prices other than p^A cannot lower utility, by the standard gains-from-trade argument. Thus,

$$w^{A*}V \geq y(p^{A*}, V) = e(p^{A*}, u_0) = \lambda e(p^A, u_0) \geq \lambda e(p^A, u_A) = \lambda w^A V, \tag{7.16}$$

where $\lambda = e(p^{A*}, u_0)/e(p^A, u_0)$.

Reversing the roles of the two countries in the above argument yields:

$$w^A V^* \geq \mu w^{A*} V^*, \tag{7.17}$$

where $\mu = e(p^A, u_1)/e(p^{A*}, u_1)$ and u_1 solves $e(p^{A*}, u_1) = y(k^A, V)$. Identical homothetic demands imply that $\lambda = 1/\mu$. Then combining (7.16) and (7.17) yields:

$$(w^{A*} - \lambda w^A)(V - V^*) \geq 0. \tag{7.18}$$

Since λ can be viewed as a measure of the foreign autarkic price level relative to the domestic, correlation (7.18) says that countries tend to have relatively low autarkic prices for those factors with relatively large endowments: the price and quantity definitions are positively correlated. When $m = 2$ the correlation becomes an identity. The assumption of identical homothetic demands is normally the key to establishing this identity, just as that assumption is now the key to establishing a correlation for the general case.

Further progress requires increasingly severe restrictions. Suppose $n \geq m$. Since commodity prices uniquely relate to factor prices once endowments v are given, write $w = f(p, v)$. Define $h(p, v) = f(p, v)(V - V^*)$, and use logic like that preceding (7.4) to obtain, for some $\bar{w} = f(\bar{p}, \bar{v})$:

$$\begin{aligned} h(p^A, V) - h(p^{A*}, V^*) = {} & (p^A - p^{A*}) f_p(\bar{p}, \bar{v})(V - V^*) \\ & + (V - V^*) f_v(\bar{p}, \bar{v})(V - V^*), \end{aligned}$$

where f_p and f_v denote the appropriate matrices of partial derivatives. Now $f_v(\bar{p}, \bar{v}) = 0$ by the factor-price equalization theorem and $f_p(\bar{p}, \bar{v}) = A^{-1}(\bar{w})$ with $A(\bar{w})$ made square, if necessary, by the arbitrary deletion of extra goods, as in subsection (7.4). Then use (7.18), with prices normalized so that $\lambda = 1$, to obtain:

$$(p^{A*} - p^A) A(\bar{w})^{-1}(V - V^*) \geq 0. \tag{7.19}$$

Thus countries tend to have a comparative advantage in goods that make intensive use of factors that are relatively abundant in a physical sense. Such conclusions in a 2×2 world usually follow an assumption of no factor intensity reversals. In the more general context the proper choice of $\bar{w}$ plays the same role: recall the earlier discussion following the derivation of (7.4). The present argument can be successively applied to subsets of goods when $n > m$, but breaks down if $m > n$. Note also the interesting point that in the price version the relevant concept of factor intensity had to do with the relative magnitudes of the elements of A, whereas the quantity version uses A^{-1}. These concepts become distinct when dimensionality exceeds two.

More detailed conclusions follow when, in addition to $n \geq m$, endowments are sufficiently similar so that free trade produces factor price equalization.[29] Let V^w denote the vector of world factor endowments and g the ratio of domestic income to world income in the free trade equilibrium. Identical homothetic tastes imply that countries consume goods in identical proportions, and factor price equalization implies that each good is produced by a single technique regardless of place of production. Thus the vector of goods consumed at home must require the factors gV^w for its production so that the factor content of the home country's trade is $M_V = gV^w - V$. A country imports its relatively scarce factors, with the degree of importation proportional to the degree of scarcity. Note that, because of factor price equalization, this result also holds in value terms, with factors valued at their free trade prices.

Thus the factor content of trade can be specified precisely. What about its commodity composition? From (7.7):

$$(V^* - V)A(w)(X^* - X) > 0,$$

where w denotes the common vector of factor prices and X^* and X denote foreign and domestic production in free trade. If the foreign economy is scaled by any scalar g and the home economy by $(1-g)$, the same method yields $[gV^* - (1-g)V]A(w)[gX^* - (1-g)X]$. Now if g is set equal to the ratio of home income to world income, in free trade, and if the countries share identical homothetic tastes, home imports $M = gX^* - (1-g)X = g(X^* + X) - X$. Thus,

$$\left[(g(V+V^*) - V\right]AM \geq 0. \tag{7.20}$$

Thus a country tends on average to import those goods which make relatively intensive use of its relatively scarce factors in a quantity sense, where a factor is scarce or abundant according to whether the home country accounts for a smaller

[29]See Travis (1964), Vanek (1968), Bertrand (1972), Williams (1977), Harkness (1978), Leamer (1980), and Ethier (1982).

or greater supply of that factor than of factors in general (evaluated at the common factor prices).

8. Odd or even: Does it matter?

Higher dimensional generalization of the 2×2 theory involves consideration both of larger numbers of goods and factors and also of unequal numbers of the two. The latter has turned out to be significant. Many results, especially among those of a "strong" sort discussed in Section 6, require an even technology: the number of goods must exactly equal the number of factors.

This would seem most unfortunate. The existence of goods and factors is a fact of technology and of nature and generally taken as exogenous by economists. It is too much to ask of the world that it accommodate itself to our theories by providing factors and goods in precisely equal numbers. Furthermore the application of theory to reality is in this case sufficiently murky to destroy most people's intuition about which body of theory is the relevant one, when odd and even technologies yield distinct conclusions. The sensitivity of our results to such an arbitrary facet of technology appears fatal.

That such a pessimistic view may not be warranted is suggested by two considerations. The general impression conveyed by the "strong" literature discussed in Section 6 was of the difficulty in breaking the bounds of "twoness", whereas the alternative approach of Section 7 established the general validity of a significant core of 2×2 results. This suggests that the important question is whether the weak results depend upon the technology being odd or even. Now recall that the weak results that were valid when $n = m$ had to be altered in only trivial ways when the number of goods exceeded the number of factors, but that the opposite circumstance of relatively more factors required substantial alteration. Thus the crucial issue would seem to be not whether the technology is even or odd, but rather whether the number of goods is at least as large as the number of factors or not.

This would appear to considerably improve our odds. Also a number of writers have argued [Travis (1972), Rader (1979)] that the case of more goods than factors is in fact the practically relevant one. But this still leaves the significance of our theory subject to an arbitrary facet of technology and nature. This is where the second consideration comes in. Two distinctions between good and factor figure prominently in the theory:

(i) factors are primary inputs and goods are outputs – the definitional distinction; and

(ii) goods are internationally traded and factors are not.

Now the relative numbers of primary inputs and outputs is indeed an arbitrary matter of technology and nature, but the relative number of international and national markets is to a significant degree determined by endogenous social and economic organization and by equilibrium prices. Should the latter aspect prove to be the decisive one, the fact that basic propositions depend upon the relative numbers of goods and factors would turn out to be no embarrassment at all: properties sensitive to the number of markets in existence are the very bread and butter of economics.

Furthermore, there is in fact every reason to believe that the crucial aspect is indeed the relative number of international markets and not the technological distinction between good and factor. A basic theme of the factor endowment theory has always been the substitutability of international commodity trade for international factor trade. And from the point of view of analytical hurdles, the introduction of international capital markets into models with $m > n$ would appear on balance a simplification rather than a complication.

This question has surprisingly been ignored in the literature, despite its evident importance and the apparent ease of approaching it.[30] A full scale assault will not be attempted now, but the most strategic points can be carried forthwith.

Consider first the factor-price equalization property, which is especially susceptible to an excess of factors over goods. Let m_{T} of the m factors be freely traded, and let m_N be internationally immobile. Consider an equilibrium:

$$p = (w_{\mathrm{T}}, w_{\mathrm{N}}) A(w_{\mathrm{T}}, w_{\mathrm{N}}), \tag{8.1}$$

$$(V_{\mathrm{T}}, V_{\mathrm{N}}) = A(w_{\mathrm{T}}, w_{\mathrm{N}}) X, \tag{8.2}$$

where w_{T} and w_{N} denote the rentals of the respective sets of factors, V_{N} the endowment of immobile factors, and V_{T} the use of tradable factors. In (8.1), p and w_{T} are determined on international markets and w_{N} internally. If w_{N} satisfies (8.1) for given p and w_{T}, define $K_{w_{\mathrm{N}}}(p, w_{\mathrm{T}})$ to be the set of all V_{N} satisfying (8.2) for non-negative X and V_{T}. An argument similar to that of Section 7 establishes that $K_{w_{\mathrm{N}}}(p, w_{\mathrm{T}}) \cap K_{w_{\mathrm{N}}^1}(p, w_{\mathrm{T}}) \neq \varnothing$ implies that $w_{\mathrm{N}} = w_{\mathrm{N}}^1$. Thus all countries with endowments of nontraded factors in $K_{w_{\mathrm{N}}}(p, w_{\mathrm{T}})$ have factor prices $(w_{\mathrm{N}}, w_{\mathrm{T}})$ if freely trading goods and traded factors at prices p and w_{T}. For this to be a significant factor-price equalization result, $K_{w_{\mathrm{N}}}(p, w_{\mathrm{T}})$ must be of dimensionality m_{N}, that is, $A(w_{\mathrm{T}}, w_{\mathrm{N}})$ must have at least m_{N} columns (linearly independent in their use of *non-traded* factors), so the dimensionality constraint is $n \geq m_{\mathrm{N}}$, or

$$n + m_{\mathrm{T}} \geq m. \tag{8.3}$$

The number of international markets must be at least as great as the number of factors.

[30] The relevant literature includes Inada (1971), Rodriguez (1975), Neary (1980), and Svensson (1982). These papers also discuss the symmetrical possibility of nontraded goods, ignored below.

The argument of Section 7 regarding real rewards goes through as before, relating commodity price changes to the rewards of *non-traded* factors. In addition one can now note how changes in the rewards of traded *factors* influence the rewards of non-traded factors. This line of inquiry reveals some interesting contrasts to the usual one, but it will not be followed now. When condition (8.3) is imposed, the earlier Rybczynski discussion when $n \geq m$ also goes through without difficulty, regarding the effects of changes in the endowments of *non-traded* factors on commodity outputs. It now becomes possible to ask how changes in the endowments of non-traded factors influence the use of traded factors, but this will not be done. In general, when (8.3) holds the earlier Stolper–Samuelson and Rybczynski results for $n \geq m$ continue to hold; the technology need be fully latticed with respect only to non-traded factors.

Thus it would appear that the crucial consideration for the higher dimensional generalization of 2×2 results is not the exogenous happenstance of the relative numbers of goods and factors, but the economic condition of the number of international markets. But note that this is not completely true, as it would be, for example, if the crucial condition had turned out to be that the number of international markets be at least as great as the number of national. Rather we have as many international markets as there are *factor* markets: the larger the number of factors relative to goods, the larger the fraction of total markets that must be international.

Return one last time to the specific-factors model for illustration. With n goods and $n+1$ factors, one factor market must be international for the model to satisfy condition (8.3). Suppose first that the intersectorally mobile factor is also internationally mobile. With internationally identical technology, the home country can compete in any commodity market on equal terms with the foreign country only if the respective specific factor receives the same reward in both countries, since the mobile factor fetches a single world price. Thus all factor prices are equalized: factor-price equalization is even more pervasive now than in the standard $n \times n$ world.

If instead one of the specific factors is internationally mobile, that sector can be operated in both countries only if the mobile factor receives the same reward in both countries and therefore, by the above argument, only if all factor prices are equalized. Thus factor-price equalization again follows as long as the single internationally mobile specific factor does not in equilibrium locate entirely in one country. This will not happen if endowments of the other n factors are sufficiently similar across the countries. Thus factor-price equalization becomes qualitatively about as inherent as in the $n \times n$ model.

Comparative statistics results are influenced by the fact that $A(w)$ is not fully latticed, a consideration that becomes more prominent once oddness is disposed of. The Stolper–Samuelson conclusions now emerge in full force in nearly every case, with the price of the mobile factor treated like a commodity price in the experiments. But the Rybczynski conclusions are pervaded by "ties": an increase in the endowment of any specific factor, holding constant all commodity prices and the reward of the internationally mobile factor, must cause the corresponding sector to expand in the same proportion as the factor (accompanied by an inflow of the mobile factor if that is internationally mobile, or an outflow of some specific factor if instead it is internationally mobile), with no change in all (or all but one) other outputs. An increase in the endowment of the internally mobile factor, with one of the specific factors mobile internationally, produces a more than proportional rise in the output corresponding to the latter, but no change at all in other outputs.

9. Concluding remarks

The elaborate and extensive structure of modern trade theory has been built on a foundation of several extreme assumptions, including that of low and even dimensionality. A large volume of theoretical work in recent decades has exposed the sensitivity of the structure to these assumptions, and at the same time extensive empirical work has claimed to demonstrate that low dimensionality, at least, is fundamentally inadequate. This has threatened to make a shambles of our theory. But, at least as regards dimensionality, the general implication of recent work, as surveyed in this chapter, seems to be more hopeful than that. Dimensionality has, to be sure, been seen to matter. But the interesting conclusion, to me, is the large extent to which the basic messages of elementary theory still come through.

They do so in two ways. First some results (the law of comparative advantage, Heckscher–Ohlin theorems, and the directional comparative-statistics predictions based on factor intensity) survive as correlations, or in an average sense. This happens to propositions that rely heavily on revealed preference logic and its analogues. Other results (Stolper–Samuelson and Rybczynski) survive in undiluted strength but only in a nonexclusive sense: they apply to some factors or goods but not necessarily to all. In all cases though the 2×2 theory has turned out to have pointed the way with a good deal of accuracy.

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