I. Algebra

1. Vector Spaces

1A. Linearity.

Definition 1.1. A *real vector space* (or just a *vector space* for short) consists of a set V, a function $V \times V \to V$ called *addition*, and a function $\mathbf{R} \times V \to V$ called *scalar multiplication*.

For $v, w \in V$, we write $v + w$ for the image of (v, w) under addition. For $\alpha \in \mathbf{R}$ and $v \in V$, we write αv for the image of (α, v) under scalar multiplication.

Addition and scalar multiplication are required to satisfy the following axioms:

i) $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$. This allows us to write expressions like $u + v + w$ and interpret them unambiguously.

ii) $u + v = v + u$ for all $u, v \in V$.

iii) There is an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

iv) For every $v \in V$ there is an element $-v$ such that $v + (-v) = 0$. We write $u - v$ as an abbreviation for $u + (-v)$.

v) $(\alpha\beta)v = \alpha(\beta v)$ for all $v \in V$ and $\alpha, \beta \in \mathbf{R}$.

vi) $\alpha(v + w) = \alpha v + \alpha w$ for all $v, w \in V$ and $\alpha \in \mathbf{R}$.

The elements of a vector space are called *vectors*.

Example 1.1.1. $\mathbf{R}^n$ is a vector space under the operations

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$

$$\alpha(a_1, \ldots, a_n) = (\alpha a_1, \ldots, \alpha a_n)$$

Definition 1.2. Let V and W be real vector spaces. A *linear transformation* is a function $f : V \to W$ such that

$$f(\alpha v_1 + v_2) = \alpha f(v_1) + f(v_2) \tag{1.2.1}$$

for all $v_1, v_2 \in V$ and $\alpha \in \mathbf{R}$. We will sometimes say "f is linear" to mean that f is a linear transformation.

Definitions 1.3. A linear transformation is an *isomorphism* if it is one-one and onto. If there is an isomorphism $f : V \to W$, we say that V is *isomorphic* to W and write $V \approx W$.

Proposition 1.4.

a) The inverse of an isomorphism is an isomorphism.

b) The composition of two isomorphisms is an isomorphism.

Corollary 1.4.1.

a) If $V \approx W$ then $W \approx V$.

b) If $U \approx V$ and $V \approx W$ then $U \approx W$.

1B. Bases and Dimension

Definition 1.5. A set of elements $\{v_i\}$ in V is a *basis* if every element of $v \in V$ can be written *uniquely* as

$$v = \sum_i \alpha_i v_i \tag{1.5.1}$$

with all but finitely many α_i equal to zero (so the sum makes sense).

Subdefinitions 1.5.2. The condition for $\{v_i\}$ to be a basis can be broken into two parts:

i) We say that the v_i *span* V if for every $v \in V$, there is *at least* one expression of the form (1.5.1) (in which case we say that v is a *linear combination* of the v_i).

ii) We say that the v_i are *linearly independent* if for every $v \in V$, there is *at most* one expression of the form (1.5.1).

Clearly, $\{v_i\}$ is a basis if and only if the v_i both span V and are linearly independent.

Definition 1.6. A vector space is *finite dimensional* if it has a finite basis. In that case, the *dimension* of the vector space is the cardinality of a basis. This definition makes sense by (1.6). The dimension of V is denoted $\dim(V)$.

Exercise 1.6.1. Prove that

$$\dim(\mathbf{R}^n) = n$$

Convention 1.7. Henceforth, except in a few cases where we explicitly state otherwise, all vector spaces in this book are assumed to be finite dimensional. This assumption will (slightly) simplify some of the notation in the proofs. All of the proofs can be modified to make this assumption unnecessary.

Proposition 1.8. Any two bases for the same vector space have the same cardinality.

Proof. Let $\{v_1, \ldots v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases for V. According to (1.5.1), there are expressions

$$v_i = \sum_{j=1}^{m} \alpha_{ij} w_j \qquad\qquad w_j = \sum_{k=1}^{n} \beta_{jk} v_k$$

Combining these gives

$$v_i = \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_{ij} \beta_{jk} v_k \qquad\qquad w_j = \sum_{k=1}^{n} \sum_{i=1}^{m} \beta_{jk} \alpha_{ki} w_i$$

From the uniqueness of the representation (1.5.1), we conclude that

$$1 = \sum_{j=1}^{m} \alpha_{ij} \beta_{ji} \qquad\qquad 1 = \sum_{k=1}^{n} \beta_{jk} \alpha_{kj}$$

which gives

$$n = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} \beta_{ji} \qquad\qquad m = \sum_{j=1}^{m} \sum_{k=1}^{n} \beta_{jk} \alpha_{kj} \tag{1.8.1}$$

But the two double summations in (1.8.1) are clearly equal, whence $n = m$.

Proposition 1.9. Let V be a vector space with basis $\{v_1, \ldots, v_n\}$. Let W be a vector space and let $w_1, \ldots, w_n$ be any elements of W. Then there is a unique linear transformation $f : V \to W$ such that $f(v_i) = w_i$.

Proof. For existence, write an arbitrary element $v \in V$ as $\sum_i \alpha_i v_i$ and define $f(v) = \sum_{i=1}^{n} \alpha_i w_i$. This is well-defined because the expression $v = \sum_i \alpha_i v_i$ is unique (see (1.5)).

For uniqueness, note again that every $v \in V$ is of the form $v = \sum_i \alpha_i v_i$, so that the value of $f(v)$ is determined by the values of the $f(v_i)$ and condition (1.2.1).

Proposition 1.10. If two vector spaces V and W have the same dimension, they are isomorphic.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for V and $\{w_1, \ldots w_n\}$ a basis for W. Use (1.9) to construct a linear transformation $f : V \to W$ such that $f(v_i) = w_i$. Check that f is one-one and onto.

Remark 1.11. It is said that the three most important rules in linear algebra are:

i) Never choose a basis.

ii) Never even *think* about choosing a basis.

iii) When you choose a basis, choose it carefully.

We will adopt the spirit of all three rules, choosing bases only when there is a compelling reason to do so and recognizing that a basis-free argument is usually more enlightening than an argument that depends on arbitrary choices.

One exception: In arguments about dimension, it is generally unavoidable to acknowledge the existence of a basis, because dimension is *defined* in terms of bases.

1C. New Spaces From Old

Definition 1.12. The *zero vector space* is the vector space with one element 0.

Definition 1.13. Given two vector spaces V and W, their *direct sum* $V \oplus W$ is the set of ordered pairs $\{(v, w) | v \in V, w \in W\}$, with addition and scalar multiplication defined "componentwise":

$$(v_1, w_1) + (v_2, w_2) = (v_1 + w_1, v_2 + w_2)$$

$$\alpha(v, w) = (\alpha v, \alpha w)$$

Definition 1.14. Let S be any set. Then the *free vector space* on S is the set of all those functions $f : S \to \mathbf{R}$ such that $\{s \in S | f(s) \neq 0\}$ is finite. Addition and scalar multiplication are defined via

$$(f+g)(s) = f(s) + g(s)$$

$$(\alpha f)(s) = \alpha f(s)$$

Notice that S is a basis for F, so, by (1.9), we can specify a linear transformation $F \to W$ by specifying the values for $f(s)$ as s ranges over S.

Note also that F is finite dimensional if and only if S is finite.

Definition 1.15. Let $U \subset V$ be vector spaces. If the addition and scalar multiplication on U are the restrictions of addition and scalar multiplication on V, then U is called a *subspace* of V.

Example 1.15.1. Let V and W be vector spaces. Then $\{(v,0)|v \in V\}$ and $\{(0,w)|w \in W\}$ are subspaces of $V \oplus W$. We will abuse notation by identifying these subspaces with V and W along the isomorphisms $(v,0) \mapsto v$ and $(0,w) \mapsto w$.

Exercise and Definition 1.15.2. Let V be a vector space and S any subset of V. Define $< S >$ to be the set of all finite sums

$$< S >= \{\sum \alpha_i s_i | \alpha_i \in \mathbf{R}, s_i \in S\}$$

Show that $< S >$ is a subspace of V. Show that if U is any subspace of V and $S \subset U$ then $< S > \subset U$. We call S the *subspace generated by* S, or the *smallest subspace of* V *containing* S.

Definition 1.16. Let $U \subset V$ be a subspace. The *quotient space* V/U is derived from V by "setting the elements of U to zero". More precisely, we define two elements of V to be *equivalent mod* U if their difference is contained in U and define V/U to be the set of equivalence classes for this relation. If $\overline{v}$ represents the equivalence class of v, we define addition and scalar multiplication by

$$\overline{v} + \overline{w} = \overline{v+w}$$

$$\alpha\overline{v} = \overline{\alpha v}$$

Proposition 1.16.1. Let $U \subset V$ be a subspace and W a vector space. Then there is a one-one correspondence between

i) Linear transformations $f : V \to W$ that map all elements of U to zero.

ii) Linear transformations $\overline{f} : V/U \to W$.

Proof. Given f, define $\overline{f}$ by $\overline{f}(\overline{v}) = f(v)$. Given $\overline{f}$, define f by $f(v) = \overline{f}(\overline{v})$. Check that these definitions give well- defined linear transformations and that the two processes are inverse to each other.

Proposition 1.17. If U is a subspace of V, then

$$\dim(V) = \dim(U) + \dim(V/U)$$

Sketch of Proof. Let $\{u_1, \dots, u_m\}$ be a basis for U and let $\{\overline{v_1}, \dots, \overline{v_n}\}$ be a basis for V/U; now show that

$$\{u_1, \dots, u_m, v_1, \dots, v_n\}$$

is a basis for V.

Corollary 1.17.1. If $U \subset V$ is a subspace then $\dim(U) \leq \dim(V)$.

Corollary 1.17.2. If V has dimension n and $u_1, \dots, u_n$ are linearly independent elements of V, then $u_1, \dots, u_n$ form a basis for V.

Proof. Let $U \subset V$ be the subspace consisting of all elements of the form $\sum_{i=1}^{n} \alpha_i u_i$. Then

by (1.17), V/U is zero-dimensional; thus $U = V$ so the u_i span V.

2. Hom and Tensor

2A. Hom

Definition 2.1. We write $\mathrm{Hom}(V, W)$ for the set of all linear transformations $f : V \to W$.

Definition 2.2. For $f, g \in \mathrm{Hom}(V, W)$, we define the *sum* $f + g \in \mathrm{Hom}(V, W)$ by the condition

$$(f+g)(v) = f(v) + g(v)$$

For $\alpha \in \mathbf{R}$ and $f \in \mathrm{Hom}(V,W)$, we define the *scalar product* $\alpha f \in \mathrm{Hom}(V,W)$ by the condition

$$(\alpha f)(v) = \alpha f(v)$$

Proposition 2.3. With addition and scalar multiplication defined as in 2.2, $\mathrm{Hom}(V,W)$ is a vector space.

Proposition 2.4.

$$\dim(\mathrm{Hom}(V,W)) = \dim(V)\dim(W)$$

Proof. Let $\{v_i\}$ be a basis for V and let $\{w_j\}$ be a basis for W. Define $f_{ij} : V \to W$ by requiring

$$f_{ij}(v_k) = \begin{cases} w_j & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \tag{2.4.1}$$

(Such a map exists and is unique by (1.9.)) Then check that $\{f_{ij}\}$ is a basis for $\mathrm{Hom}(V,W)$.

Corollary 2.4.1. $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m) \approx \mathbf{R}^{nm}$.

Proof. Use (2.4), (1.17), and (1.10).

Definition 2.5. The *dual* of V is the vector space $V^* = \mathrm{Hom}(V, \mathbf{R})$.

Proposition 2.6. The vector spaces V and V^* are isomorphic.

Proof. By (2.4), $\dim(V) = \dim(V^*)$. Now use (1.10).

Remarks 2.6.1. Given a basis $\{v_1, \ldots, v_n\}$ for V, we can construct a basis $\{v_1^*, \ldots, v_n^*\}$ for V^* by defining

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The basis $\{v_i^*\}$ is called the *dual basis* to the basis $\{v_i\}$.

The construction of the dual basis is a special case of the construction in the proof of 2.4; take $W = \mathbf{R}$ with the single basis element $w_1 = 1$; then the v_i^* of (2.6.1.1) is equal to the f_{i1} of (2.4.1).

Therefore a choice of basis $\{v_i\}$ yields an explicit isomorphism $\phi : V \to V^*$, namely

$$\phi(v_i) = v_i^* \tag{2.6.1.2}$$

((2.6.1.2) defines ϕ uniquely by (1.9).)

Note that this isomorphism depends on the choice of basis; if we had chosen a different basis for V, we would have constructed a different isomorphism.

There is no good way to select a "preferred" isomorphism from V to V^*; we will revisit this issue in Section 5 of this chapter.

Remark 2.6.1.3 The element v_i^* of the dual basis depends not just on v_i but on all the v_j. If $\{w_1, \ldots, w_n\}$ is another basis and $v_1 = w_1$, it does *not* follow that $v_1^* = w_1^*$.

Definition 2.7. The *double dual* of V is the vector space $V^{**} = (V^*)^*$.

2.8. Applying (2.6) twice and invoking (1.4), we can construct an isomorphism $V \to V^{**}$. But there is a simpler, basis- independent construction. Map

$$\phi : V \to V^{**}$$

by

$$\phi(v)(g) = g(v)$$

and check that this map is one-one and onto.

2B. Multilinearity

Definition 2.9. Let V, W and U be vector spaces. Then a *bilinear map*

$$f : V \times W \to U$$

is a function satisfying the following axioms for $v_i \in V$, $w_i \in W$ and $\alpha \in \mathbf{R}$:

i) $f(\alpha v_1 + v_2, w) = \alpha f(v_1, w) + f(v_2, w)$

ii) $f(v, \alpha w_1 + w_2) = f(v, w_1) + \alpha f(v, w_2)$

Remark 2.9.1. You can think of multilinearity as "linearity in each variable separately". More precisely, we can restate conditions (2.9(i)) and (2.9(ii)) as follows:

i′) For each fixed $v \in V$, the map

$$f_v : W \to U$$

defined by

$$f_v(w) = f(v, w)$$

is linear.

ii′) For each fixed $w \in W$, the map

$$f_w : V \to U$$

defined by

$$f_w(v) = f(v, w)$$

is linear.

Remark 2.2.1.2. As sets, $V \times W$ is the same thing as $V \oplus W$, so you might be tempted to think of a bilinear map as a function

$$f : V \oplus W \to U$$

But it is important to realize that this map is *not* in general a linear transformation. Here's why: Applying axioms 2.9(i) and 2.9(ii), we find that for a bilinear map f we have

$$f(v_1 + v_2, w_1 + w_2) = f(v_1, w_1) + f(v_1, w_2) + f(v_2, w_1) + f(w_1, w_2)$$

whereas for a linear transformation f we have

$$f(v_1 + v_2, w_1 + w_2) = f(v_1, w_1) + f(v_2, w_2)$$

which is not at all the same thing.

We will want to adopt the convention that whenever we write down a map between vector spaces, it is assumed to be linear. Therefore, when f is a bilinear map, we will think of its domain as the *set* of ordered pairs $V \times W$, rather than the *vector space* of ordered pairs $V \oplus W$.

Definition 2.10. We generalize the notion of bilinearity as follows: A map

$$f : V_1 \times V_2 \times \cdots \times V_n \to W$$

is called *multilinear* if it is linear in each variable separately. More precisely, the condition is that if we fix elements in $n-1$ of the vector spaces $V_1, \ldots V_n$, then the induced map on the remaining V_i is linear.

2C. Tensor Products

2.11. The *tensor product* of two vector spaces V and W is a vector space $V \otimes W$ which is the recipient of a "universal" multilinear map

$$t : V \times W \to V \otimes W \tag{2.11.1}$$

"Universal" means that any multilinear map with domain $V \times W$ can be factored uniquely as a composition $g \circ t$ where $t : V \otimes W \to U$ is a *linear* map.

To prove that the tensor product exists, we will construct it explicitly.

Definition 2.12. We define the *tensor product* $V \otimes W$ as follows.

First, let $S = V \times W$, thought of as a set, not as a vector space.

Next, let E be the free vector space on the set S. Thus a typical element of E is a linear combination

$$\sum_{i=1}^{m} \alpha_i (v_i, w_i)$$

with $\alpha_i \in \mathbf{R}$, $v_i \in V$, $w_i \in W$. (E is in general not finite dimensional.)

Next, let $F \subset E$ be the smallest subspace containing all expressions that have either of the following forms:

$$(\alpha v_1 + v_2, w) - \alpha(v_1, w) - (v_2, w) \tag{2.12.1}$$

$$(v, \alpha w_1 + w_2) - \alpha(v, w_1) - (v, w_2) \tag{2.12.2}$$

(That is, F is the subspace generated by the above elements; see (1.15.2).)

Finally, define the *tensor product* $V \otimes W = E/F$.

We write $v \otimes w$ for the image of (v, w) in $V \otimes W$. Thus a typical element of $V \otimes W$ can be written

$$\sum_{i=1}^{m} \alpha_i (v_i \otimes w_i) \tag{2.12.3}$$

though this expression is not unique.

Proposition 2.13 If $\{v_i\}$ is a basis for V and $\{w_j\}$ is a basis for W, then $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$. Therefore

$$\dim(V \otimes W) = \dim(V)\dim(W)$$

Remark 2.13.1. Not every element of $V \otimes W$ is of the form $v \otimes w$. But to describe a linear transformation $f : V \otimes W \to U$, it is enough to give a formula for $f(v \otimes w)$; according to (2.13) and (1.9) such a formula suffices to determine f.

However, the formula for $f(v \otimes w)$ cannot be specified arbitrarily. One can certainly define a map on the free vector space E by specifying $f(v, w)$ arbitrarily for each v and w, but in order for f to induce a well-defined map on the tensor product, it must (according to (1.16.1)) vanish on terms of the form (2.12.1) and (2.12.2). In other words, a formula for $f(v \otimes w)$ defines a linear transformation on $V \otimes W$ if and only if it satisfies the two conditions:

$$f((\alpha v_1 + v_2) \otimes w) = \alpha f(v_1 \otimes w) + f(v_2 \otimes w) \tag{2.13.1.1}$$

$$f(v \otimes (\alpha w_1 + w_2)) = \alpha f(v \otimes w_1) + f(v \otimes w_2) \tag{2.13.1.2}$$

Definition 2.14. Define

$$\phi_{V,W} : V \times W \to V \otimes W$$

by

$$(v, w) \mapsto v \otimes w$$

Proposition 2.15. The map $\phi_{V,W}$ is bilinear.

Proposition 2.16. Let

$$f : V \times W \to U$$

be a bilinear map. Define

$$g : V \otimes W \to U$$

by

$$g(v \otimes w) = f(v, w)$$

Then g is linear.

Theorem 2.17. There is a one- one correspondence between bilinear maps

$$f : V \times W \to U$$

and linear maps

$$g : V \otimes W \to U$$

Proof. The correspondence in one direction is given by 2.3.6. The correspondence in the other direction is given by $g \mapsto g \circ \phi_{V,W}$ (with $\phi_{V,W}$ as in 2.14).

Remark 2.17.1. Theorem (2.17) fulfills the promises made in (2.11).

Example and Definition 2.18. Consider the map

$$\begin{array}{ccc} V^* \times V & \to & \mathbf{R} \\ f \otimes v & \mapsto & f(v) \end{array} \tag{2.18.1}$$

The map (2.18.1) is easily verified to be bilinear and so yields a well-defined map, called the *trace map*

$$\begin{array}{ccc} V^* \otimes V & \to & \mathbf{R} \\ f \otimes v & \mapsto & f(v) \end{array} \tag{2.18.2}$$

2.19. All of the above can be generalized to tensor products of more than two vector spaces. Given $V_1, \ldots V_n$, we take E to be the free vector space on the set $V_1 \times \ldots \times V_n$, we take $F \subset E$ to be generated by all of those expressions (analogous to (2.12.1) and (2.12.2)) which must be mapped to zero by any bilinear map, and set

$$V_1 \otimes \cdots \otimes V_n = E/F$$

Exercise 2.19.1. Prove in detail that there is a one-one correspondence between multilinear maps

$$V_1 \times \cdots \times V_n \to U$$

and linear maps

$$V_1 \otimes \cdots \otimes V_n \to U$$

3. Functors

3A. Covariant Functors

3.1. Definition. A *covariant functor* F is a rule that associates to every vector space V a vector space $F(V)$, and to every linear transformation $f : V \to W$ a linear transformation $F(f) : F(V) \to F(W)$, subject to the following axioms:

i) If 1_V is the identity map on V, then $F(1_V) = 1_{F(V)}$.

ii) Given maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

we have

$$F(g \circ f) = F(g) \circ F(f)$$

3.1.1. We will use the word *functor* as an abbreviation for the phrase *covariant functor*.

3.2. Examples.

Example 3.2.1. (The Identity Functor). For every vector space V, let $F(V) = V$, and for every linear transformation, let $F(f) = f$. Then F is easily seen to be a functor.

Example 3.2.2 (Constant functors). Let U be a fixed vector space. For every vector space V, put $F(V) = U$, and for every map $f : V \to W$, put $F(f) = 1_U$.

Exercise 3.2.2.1. Check that F is a functor.

Example 3.2.3 (Hom Functors). Let U be a fixed vector space. For every vector space V, put $F(V) = \mathrm{Hom}(U, V)$. For every map $f : V \to W$, and for every $g \in F(V) = \mathrm{Hom}(U, V)$, put $F(f)(g) = f \circ g \in \mathrm{Hom}(U, W) = F(W)$. We will denote the functor F by the symbol $\mathrm{Hom}(U, -)$.

Excercise 3.2.3.1. Check that $\mathrm{Hom}(U, -)$ is a functor.

Example 3.2.4. (Tensor Functors). Let U be a fixed vector space. For every vector space V, put $F(V) = U \otimes V$. Given $f : V \to W$, define $F(f) : U \otimes V \to U \otimes W$ by the condition

$$F(f)(u \otimes v) = u \otimes f(v)$$

To check that this is a well-defined linear map, you can verify conditions (2.13.1.1) and (2.13.1.2). Alternatively (and more easily), first prove that the map

$$(u, v) \mapsto u \otimes f(v)$$

is bilinear on $U \times V$ and then use (2.16).

We will denote the functor F by the symbol $U \otimes -$.

Example 3.2.5. Let U be a fixed vector space. For every vector space V, put $G(V) = V \otimes U$. Given $f : V \to W$, define

$$G(f)(v \otimes u) = f(v) \otimes u$$

We will denote the functor F by the symbol $- \otimes U$.

3B. Contravariant Functors

A contravariant functor is like a covariant functor except that it reverses the directions of arrows. More precisely:

Definition 3.3. A *contravariant functor* F is a rule that associates to every vector space V a vector space $F(V)$, and to every linear transformation $f : V \to W$ a linear transformation $F(f) : F(W) \to F(V)$, subject to the following axioms:

i) If 1_V is the identity map on V, then $F(1_V) = 1_{F(V)}$.

ii) Given maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

we have

$$F(g \circ f) = F(f) \circ F(g)$$

3.3.1. Now that we have defined both covariant and contravariant functors, the single word *functor* can refer to either, depending on context.

3.4. Examples.

Example 3.4.1 (Constant functors). The constant functor of (3.2.1) is both covariant and contravariant.

Example 3.4.2 (Contravariant Hom Functors). Let U be a fixed vector space. For every vector space V, put $F(V) = \mathrm{Hom}(V, U)$. For every map $f : V \to W$, and for every $g \in F(W) = \mathrm{Hom}(W, U)$, put $F(f)(g) = g \circ f \in \mathrm{Hom}(V, U) = F(V)$. We will denote the functor F by the symbol $\mathrm{Hom}(-, U)$.

Excercise 3.4.2.1. Check that $\mathrm{Hom}(-, U)$ is a functor.

Example 3.4.3. (The Dual Functor). Specializing (3.4.2) to the case where $U = \mathbf{R}$, we get the *dual functor*, which maps V to its dual space $V^* = \mathrm{Hom}(V, \mathbf{R})$.

Example 3.4.4. (The Double Dual Functor) Applying the dual functor twice we get a *covariant* functor that takes V to its double dual V^{**} and takes the map $f : V \to W$ to the map

$$f^{**} : V^{**} \to W^{**}$$

given by

$$f^{**}(g)(h) = g(h \circ f) \tag{3.4.4.1}$$

Exercise 3.4.4.2 In (3.4.4.1), what are the domains and codomains of g and h? In what space do the left and right sides of the equation live? Verify that iterating the construction of 3.4.3 really gives equation (3.4.4.1).

3C. Multifunctors

From (3.2.3) and (3.4.2), there are functors

$$W \mapsto \mathrm{Hom}(V,W) \quad \text{(for fixed } V\text{)} \qquad \text{and} \qquad V \mapsto \mathrm{Hom}(V,W) \quad \text{(for fixed } W\text{)}$$

It is natural, then, to think of $\mathrm{Hom}(V,W)$ as a sort of "functor of two variables". A functor of many variables is called a multifunctor. We begin by making this notion precise.

Definition 3.5. A *multifunctor* F of type (p,q) is a rule that

i) takes as input an ordered p- tuple of vector spaces $(V_1,\dots,V_p)$ and an ordered q-tuple of vector spaces $(V_1',\dots,V_q')$, and gives as output a single vector space $F(V_1,\dots,V_p,V_1',\dots,V_q')$.

ii) takes as input an ordered p- tuple of linear transformations $(f_i : V_i \to W_i)$ and an ordered q-tuple of linear transformations $(f_j' : U_j' \to V_j')$, and produces as output a linear transformation

$$F(f_1,\dots,f_p,f_1',\dots,f_q') : F(V_1,\dots,V_p,V_1',\dots,V_q') \to F(W_1,\dots,W_p,U_1',\dots,U_q')$$

subject to the following axioms:

i) If 1_{V_i} and $1_{V_i'}$ are the identity maps on V_i and V_i', then

$$F(1_{V_1},\dots,1_{V_p},1_{V_1'},\dots,1_{V_q'}) = 1_{F(V_1,\dots,V_p,V_1',\dots,V_q')}$$

(ii) Given maps

$$U_i \xrightarrow{f_i} V_i \xrightarrow{g_i} W_i$$
$$U_i' \xrightarrow{f_i'} V_i' \xrightarrow{g_i'} W_i'$$

we have

$$F(g_1f_1,\dots,g_pf_p,g_1'f_1',\dots,g_q'f_q') = F(g_1,\dots,g_p,f_1',\dots,f_q')\circ F(f_1,\dots,f_p,g_1',\dots,g_q')$$

In some cases (cf. 3.6.1 below) it will be inconvenient to list covariant variables first and contravariant variables second. We will allow ourselves to list the variables in any order as long as we specify which are covariant and which are contravariant.

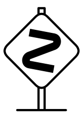
Remark 3.5.1. Given a multifunctor F, you can "fix" all but one index to get an ordinary functor $\hat{F}$. More precisely, choose fixed vector spaces to go into $p+q-1$ of the $p+q$ slots. To evaluate $\hat{F}(V)$ insert V in the remaining slot and evaluate F. If f is a linear transformation, evaluate $\hat{F}(f)$ by inserting f in the "free" slot and identity maps on the fixed vector spaces in the "fixed" slots.

Definition 3.5.2. If $p+q=2$, then a multifunctor of type (p,q) is also called a *bifunctor*.

3.6.2. Examples.

Example 3.6.1 (Hom as a bifunctor). We can combine the contravariant and covariant Hom functors (3.2.3) and (3.4.2) into a single type-$(1,1)$ multifunctor $\mathrm{Hom}(-,-)$, which maps (U,V) to $\mathrm{Hom}(U,V)$. The action on linear transformations is dictated by the requirement that the Hom bifunctor should restrict to the previously defined Hom functors (3.2.3) and (3.4.2) when either variable is fixed.

Exercise 3.6.1.2. Check that $\mathrm{Hom}(-,-)$ is a bifunctor.



Remark 3.6.1.3. Do not be misled by this example. You cannot in general define a bifunctor by requiring it to restrict to previously defined functors when either variable is fixed; without further checking, there is no guarantee that the axioms of (3.5) will be satisfied.

Example 3.6.2. (Tensor as a bifunctor). We can combine the two covariant tensor functors (3.2.4) and (3.2.5) into a type $(2,0)$-bifunctor that maps (U,V) to $U\otimes V$.

Exercise 3.6.2.1. Explicitly describe the action of this bifunctor on a pair of linear transformations, and check that it is indeed a bifunctor.

Exercise 3.6.2.2. Use example (2.19.1) to construct a multifunctor of type $(n, 0)$ that takes $(V_1, \ldots, V_n)$ to $V_1 \otimes \ldots \otimes V_n$.

Example 3.6.3. We can compose functors with multifunctors in obvious ways to get more multifunctors. For example we could define a type-$(2, 1)$ multifunctor

$$F : (U, V, W) \mapsto \mathrm{Hom}(U, V \otimes W) \tag{3.6.3.1}$$

In such cases, we will generally specify the action on vector spaces and leave it to the reader to infer the action on linear transformations, which will always (unless specified otherwise) be dictated by compatibility with functors already defined.

Exercise 3.6.3.2. For the multifunctor F of (3.6.3.1), write down an explicit description of the action of F on a triple of linear transformations.

3D. Composed Functors.

Definition 3.7. Let F and G be functors, each either covariant or contravariant. We define the *composed functor* $G \circ F$ by setting $(G \circ F)(V) = G(F(V))$ and $(G \circ F)(f) = G(F(f))$ for every vector space V and every linear transformation f. (You should of course check that $G \circ F$ is really a functor).

Proposition 3.7.1. If F and G are either both covariant or both contravariant, then $G \circ F$ is a covariant functor. If one of F and G is covariant and the other is contravariant, then $G \circ F$ is a contravariant functor.

Definition 3.8. We generalize (3.7). Let $F_1, \ldots F_n$ be functors (each either covariant or contravariant) and let G be a multifunctor of type (p, q) where $p+q = n$. Let F represent the n-tuple $(F_1, \ldots, F_n)$. Then set $(G \circ F)(V_1, \ldots, V_n) = G(F_1(V_1), \ldots, F_n(V_n))$ and $(G \circ F)(f_1, \ldots, f_n) = G(F_1(f_1), \ldots, F_n(f_n))$ for vector spaces V_i and linear transformations f_i. The multifunctor $G \circ F$ is called the *composition* of G with F.

4. Naturality

4A. Introduction to Naturality.

4.1. To prove that a vector space V is isomorphic to its dual V^*, we first choose a basis for V and then proceed as in (2.6.1). The isomorphism we end up with depends on the basis we start with, and is in that sense arbitrary. On the other hand, to prove that V is isomorphic to its double dual V^{**}, there is no need to choose a basis; (2.8) constructs an isomorphism that does not depend on any arbitrary choices.

To a rough approximation, the concept of naturality is intended to distinguish between isomorphisms that seem "arbitrary" and isomorphisms that seem "natural". Indeed, many textbooks introduce the concept of naturality by asserting that any vector space V is naturally isomorphic to its own double dual but not to its dual. But this statement is misleading. The concept of naturality does not apply to individual vector spaces and individual isomorphisms; it applies to functors.

We shouldn't ask whether V is naturally isomorphic to V^*; instead we should ask whether the identity functor (which takes V to V) is isomorphic to the dual functor (which takes V to V^*). According to the definitions we will introduce, this amounts to asking whether we can *simultaneously* choose, for *every* vector space V, an isomorphism $\phi_V : V \to V^*$ in such a way that all the choices are consistent with the action of the dual functor. (It will turn out that the answer to this question is no, but if the dual is replaced by the double dual, then the answer is yes.) If the choices can be made consistently, we will call them "natural"; if not, not. Our next goal is to make this idea precise.

4.2. Just as a functor takes vector spaces to vector spaces, a *natural transformation* takes functors to functors. Suppose F and G are covariant functors. Then a natural transformation

$$\phi : F \Rightarrow G$$

consists of a choice, for every vector space V, of a linear transformation

$$\phi_V : F(V) \to G(V)$$

These choices are subject to the additional condition that ϕ should be compatible with the actions of F and G on linear transformations. When F and G are both covariant, this means that for any linear transformation $f : V \to W$, the following diagram must commute:

$$\begin{array}{ccc} F(V) & \xrightarrow{\phi_V} & G(V) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(W) & \xrightarrow{\phi_W} & G(W) \end{array} \tag{4.2.1}$$

If F and G are contravariant, diagram (4.2.1) makes no sense, so we require commutativity of the following diagram instead:

$$\begin{array}{ccc} F(V) & \xrightarrow{\phi_V} & G(V) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(W) & \xrightarrow{\phi_W} & G(W) \end{array} \tag{4.2.2}$$

Definition 4.3. A natural transformation is a *natural isomorphism* if ϕ_V is an isomorphism for every V. If there exists a natural isomorphism $\phi : F \Rightarrow G$, we will say that the functors F and G are *naturally isomorphic*.

Example 4.3.1 The maps

$$\phi_V : V \to \mathrm{Hom}(\mathbf{R}, V)$$

given by

$$\phi_V(v)(\alpha) = \alpha v$$

constitute a natural isomorphism from the identity functor (3.2.1) to the functor $\mathrm{Hom}(\mathbf{R}, -)$ (3.2.3).

Example 4.3.2 The maps

$$\phi_V : \mathbf{R} \otimes V \to V$$

given by

$$\phi_V(\alpha \otimes v) = \alpha v$$

constitute a natural isomorphism from the functor $\mathbf{R} \otimes -$ (3.2.4) to the identity functor (3.2.1).

Example 4.3.3. The maps

$$\phi_V : U \otimes V \to V \otimes U$$

given by

$$\phi_V(u \otimes v) = v \otimes u$$

constitute a natural isomorphism from the functor $U \otimes -$ (3.2.4) to the functor $- \otimes U$ (3.2.5).

4B. Duality

4.4. Let F be the identity functor (3.2.1), so that $F(V) = V$, and let G be the double dual functor (3.4.4), so that $G(V) = V^{**}$. For each vector space V, let

$$\phi_V : V \to V^{**}$$

be the isomorphism described in (2.8). You should check that ϕ is a natural isomorphism by verifying that for any linear transformation $f : V \to W$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi_V} & V^{**} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f^{**}} \\ W & \xrightarrow{\phi_W} & G(W) \end{array}$$

This proves:

Proposition 4.4.1. The identity functor is naturally isomorphic to the double dual functor.

4.5. (This subsection involves subtle issues that will not arise again in this book, so feel free to skip it.)

Let F be the identity functor (3.2.1), so that $F(V) = V$, and let G be the dual functor (3.4.3), so that $G(V) = V^*$. Then F is covariant and G is contravariant, so the notion of a natural transformation $\phi : F \Rightarrow G$ is undefined. Therefore it makes no sense even to *ask* whether F and G are naturally isomorphic.

In one sense, we've just proved that there is no natural isomorphism from the identity functor to the dual functor. In another sense, we've just proved that our definition of naturality is utterly incapable of addressing the issues for which it was created in the first place. So let's temporarily broaden our definition of naturality in order to give these functors a fighting chance.

Because F is covariant and G is contravariant, the natural generalization of (4.2.1) and (4.2.2) is

$$\begin{array}{ccc} F(V) & \xrightarrow{\phi_V} & G(V) \\ {\scriptstyle F(f)}\downarrow & & \uparrow{\scriptstyle G(f)} \\ F(W) & \xrightarrow{\phi_W} & G(W) \end{array} \tag{4.5.1}$$

and we could require that (4.5.1) commute for any choice of $f : V \to W$. But this is asking far too much; by taking W to be the zero vector space, we could immediately conclude that ϕ_V is the zero map for all V; surely, then, ϕ_V has no chance to be an isomorphism (except when V itself is the zero space). Once again, we've proved that F and G are not naturally isomorphic, but once again we've done it by concocting a ridiculously restrictive definition of naturality.

So let's loosen it up a little further: We require (4.5.1) to commute only when f is an isomorphism. In that case, it's not hard to prove that $F(f)$ and $G(f)$ are isomorphisms, so $G(f) \circ \phi_W \circ F(f)$ is an isomorphism and has at least some chance of being equal to ϕ_V.

Now we can say that the identity functor and the dual functor are naturally isomorphic if there are isomorphisms $\phi_V : V \to V^*$ insuring that whenever $f : V \to W$ is an isomorphism, the following diagram will commute:

$$\begin{array}{ccc} V & \xrightarrow{\phi_V} & V^* \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^*} \\ W & \xrightarrow{\phi_W} & W^* \end{array}$$

But despite all our concessions, this remains an impossible condition to satisfy. Given $v \in V$, the commutativity of the diagram requires that

$$\phi_V(v) = f^*((\phi_W \circ f)(v)) = (\phi_W \circ f)(v) \circ f \tag{4.5.2}$$

Each side of (4.5.2) is a map from V to $\mathbf{R}$ and so can be evaluated at an arbitrary element $v' \in V$. Thus we can write

$$\phi_V(v)(v') = (\phi_W \circ f)(v)(f(v')) \tag{4.5.3}$$

If V has dimension at least 2, we can choose v' so that v and v' are members of a basis. We can then use (1.9) to construct an isomorphism $g : V \to W$ such that $g(v) = f(v)$ and $g(v') = 2f(v')$. Replacing f with g leaves the left side of (4.5.3) fixed while multiplying the right side by 2. Therefore (4.5.3) cannot remain true when f is replaced by an arbitrary isomorphism.

Therefore, even under our temporarily liberalized definition of naturality, the identity functor and the dual functor are not naturally isomorphic.

Example 4.6. (Like (4.5), this subsection involves subtle issues that will not arise elsewhere in this book.)

This example will demonstrate the importance of viewing natural isomorphism as a relation between *functors* rather than as a relation between *vector spaces*.

For every vector space V, choose an arbitrary isomorphism $\phi_V : V \to V^*$. Define a covariant functor G as follows:

For every vector space V, $G(V) = V^*$. For every linear transformation $f : V \to W$, $G(f) = \phi_W \circ f \circ \phi_V^{-1}$.

Then it is easy to check that ϕ is a natural isomorphism from the identity functor to G.

Thus it *is* possible to construct isomorphisms $V \to V^*$ that are natural with respect to an appropriate functor. But G, of course, is not the dual functor because it behaves differently on linear transformations.

4C. Naturality and Multifunctors

Definition 4.7. Let F and G be multifunctors of type (p, q). Then a *natural transformation* $\phi : F \Rightarrow G$ consists of a choice, for each p- tuple of vector spaces $V_1, \ldots, V_p$ and each q-tuple of vector spaces $V'_1, \ldots, V'_q$, of a linear transformation

$$\phi_{(V_1,\ldots,V_p,V'_1,\ldots,V'_q)} : F(V_1, \ldots, V_p, V'_1, \ldots, V'_q) \to G(V_1, \ldots, V_p, V'_1, \ldots, V'_q)$$

such that if any $p+q-1$ variables are held fixed, making F and G functors in the remaining variable as in (3.5.1), then ϕ becomes a natural transformation of functors.

A natural transformation is a *natural isomorphism* if each of the maps $\phi(V_1, \ldots, V_p, V_1', \ldots, V_q')$ is an isomorphism of vector spaces. In that case we say that F and G are *naturally isomorphic* as multifunctors.

Example 4.8. Consider the multifunctors $F(U,V) = U \otimes V$ and $G(U,V) = V \otimes U$ from (3.6.2). (As discussed in (3.2.3), we will abbreviate the description of multifunctors by omitting the action on linear transformations.) Then F and G are naturally isomorphic via

$$\phi_{(U,V)} : u_i \otimes v_i \mapsto v_i \otimes u_i$$

(This defines a linear transformation by (2.13.1).)

Abuse of Language 4.8.1. We will summarize the natural isomorphism between F and G by saying

$$U \otimes V \textit{ is naturally isomorphic to } V \otimes U$$

This statement replaces the functors F and G with the values of those functors on particular spaces. We will make such statments only when it is clear which multifunctors are lurking in the background.

Example 4.9. ("Natural associativity of the tensor product".) Let U, V and W be be vector spaces. There are natural isomorphisms

$$(U \otimes V) \otimes W \approx U \otimes V \otimes W \approx U \otimes (V \otimes W) \tag{4.9.1}$$

Here the symbol $(U \otimes V) \otimes W$ represents the composed functor $(U, V, W) \mapsto F(F(U,V), W)$ where F is the functor of (3.2.4); similarly $U \otimes (V \otimes W)$ means $F(U, F(V,W))$. $U \otimes V \otimes W$ represents the functor of (3.6.2.1).

We will use these natural isomorphisms to identify the three functors in (4.9.1) and will therefore write $U \otimes V \otimes W$ to represent any one of them. Likewise for tensor products of more than three vector spaces.

Example 4.10. Let U, V and W be vector spaces. There are inverse isomorphisms

$$\mathrm{Hom}(U \otimes V, W) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathrm{Hom}(U, \mathrm{Hom}(V, W)) \tag{4.10.1}$$

given by

$$\phi(f)(u)(v) = f(u \otimes v)$$

and

$$\psi(g)(u \otimes v) = g(u)(v)$$

(We have suppressed subscripts of (U, V, W) on both ϕ and ψ.) These maps fit together to give a natural isomorphism between the two sides of (4.10.1).

Example 4.11. Let U and V be vector spaces. There is an isomorphism

$$\phi_{(U,V)} : U^* \otimes V^* \to (U \otimes V)^* \tag{4.11.1}$$

given by

$$\phi_{(U,V)}(f \otimes g)(u \otimes v) = f(u)g(v)$$

The maps $\phi_{(U,V)}$ form a natural isomorphism between the two sides of (4.11.1).

Proof. We have to check that $\phi_{(U,V)}$ is an isomorphism and that it is natural.

Let $\{u_i\}$ a basis for U and $\{u_i^*\}$ the dual basis for U^* constructed in (2.6.1). Similarly, let $\{v_i\}$ be a basis for V and $\{v_j^*\}$ the dual basis.

Then $\{u_i \otimes v_j\}$ is a basis for $U \otimes V$ (2.13) and $\{(u_i \otimes v_j)^*\}$ is the dual basis for $(U \otimes V)^*$. Using (1.9), we can construct a linear transformation

$$\psi_{(U,V)} : (U \otimes V)^* \to U^* \otimes V^*$$

by

$$\psi_{(U,V)}((u_i \otimes v_j)^*) = u_i^* \otimes v_j^*$$

Now check that $\psi_{(U,V)}$ is an inverse for $\phi_{(U,V)}$, which shows that $\phi_{(U,V)}$ is an isomorphism.

Finally, to check naturality, let $f : U \to X$ and $g : V \to Y$ be linear transformations. We need to check commutativity of the diagram

$$\begin{array}{ccc} X^* \otimes Y^* & \xrightarrow{\phi_{(X,Y)}} & (X \otimes Y)^* \\ {\scriptstyle f^* \otimes g^*}\downarrow & & \downarrow{\scriptstyle (f \otimes g)^*} \\ U^* \otimes V^* & \xrightarrow{\phi_{(U,V)}} & (U \otimes V)^* \end{array}$$

for which check that if $x^* \in X^*$ and $y^* \in Y^*$ is sent around the diagram in either direction, it lands on the map

$$u \otimes v \mapsto x^*(f(u))y^*(g(v))$$

4.12. List of Natural Isomorphisms. For functors of one or two variables, the most important natural isomorphisms are "double duality" (4.4); "natural commutativity of the tensor product" (4.8), the "adjunction isomorphism" relating Hom and Tensor (4.10) and the "distributivity of * over Tensor" (4.11), as well as the natural isomorphisms (4.3.1) and (4.3.2). Here we gather these in one place and give them consecutive numbers for later reference:

$$V \approx V^{**} \tag{4.12.1}$$

$$V \otimes W \approx W \otimes V \tag{4.12.2}$$

$$\mathrm{Hom}(U \otimes V, W) \approx \mathrm{Hom}(U, \mathrm{Hom}(V, W)) \tag{4.12.3}$$

$$V^* \otimes W^* \approx (V \otimes W)^* \tag{4.12.4}$$

$$\mathrm{Hom}(\mathbf{R}, V) \approx V \tag{4.12.5}$$

$$\mathbf{R} \otimes V \approx V \tag{4.12.6}$$

4.13. Exercises.

Exercise 4.13.1. Use (4.12) to show that the following functors of type $(0, 2)$ are all naturally isomorphic:

a) $V^* \otimes W^*$	b) $(V \otimes W)^*$	c) $\mathrm{Hom}(V, W^*)$	d) $\mathrm{Hom}(V^*, W)^*$
e) $W^* \otimes V^*$	f) $(W \otimes V)^*$	g) $\mathrm{Hom}(W, V^*)$	h) $\mathrm{Hom}(W^*, V)^*$

Exercise 4.13.2. List eight multifunctors of type $(2, 0)$ that are naturally isomorphic to $V \otimes W$.

Exercises 4.13.3.

i) Use (4.10), (4.11), and (4.4.1) to construct a natural isomorphism

$$\phi_{UV} : \mathrm{Hom}(U, V) \to U^* \otimes V \tag{4.14.1}$$

ii) Construct a natural isomorphism

$$\psi_{UV} : \mathrm{Hom}(U, V) \to \mathrm{Hom}(V, U)^*$$

iii) Show that for $h : U \to U$, and for the Trace map defined in (2.18), we have

$$\mathrm{Trace}(\phi_{U,U}(h)) = \psi_{U,U}(h)(1_U)$$

iv) Prove that

$$\psi_{U,V}(f)(g) = \mathrm{Trace}(g \circ f)$$

5. Tensor Algebra.

Definition 5.1. Given natural numbers p and q, define a multifunctor $\mathcal{T}^{p,q}$ by

$$\mathcal{T}^{p,q}(V_1, \ldots, V_p, W_1, \ldots, W_q) \quad = \quad V_1 \otimes \ldots \otimes V_p \otimes W_1^* \otimes \ldots \otimes W_q^*$$

Remark 5.1.1. Repeated use of the isomorphisms in (4.12) yields a variety of multifunctors naturally isomorphic $\mathcal{T}^{p,q}$; for example, (4.13.1) gives eight bifunctors naturally isomorphic to $\mathcal{T}^{0,2}$.

Exercise 5.1.1.1. Formulate a precise version of the following statement and then prove it: Any multifunctor of type (p, q) that is a composition of Hom, Tensor, and Dual functors must be naturally isomorphic to $\mathcal{T}^{p,q}$.

Definition 5.2. Given a vector space V, define the space of (p, q)-*tensors* over V by

$$T^{p,q}(V) \quad = \quad \mathcal{T}^{p,q}(V, \ldots, V, V \ldots V) \quad = \quad V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$$

where there are p copies of V and q copies of V^* on the right-hand side.

For $p = q = 0$, put $T^{0,0}(V) = \mathbf{R}$.

Remark 5.2.1. Although $\mathcal{T}^{p,q}$ is a multifunctor, $T^{p,q}$ is not a functor (except when $p = 0$ or $q = 0$). Given an arbitrary linear transformation $f : V \to W$, there is no obvious way to construct an associated linear transformation $T^{p,q}(f) : T^{p,q}(V) \to T^{p,q}(W)$.

The exception is when f is an isomorphism, in which case we can define $T^{p,q}(f)$ to act like f on the first p factors and like $(f^*)^{-1}$ on the remaining q factors. Thus, although $T^{p,q}$ is not a full-fledged functor, we can still think of it as "functorial for isomorphisms".

Remark 5.2.2. Many books define the space of (p,q)-tensors as

$$(T^{p,q})'(V) \quad = \quad (V^* \otimes \ldots \otimes V^* \otimes V \otimes \ldots \otimes V)^*$$

with p copies of V^* and q copies of V. But repeated application of (4.12.4) provides a natural isomorphism between $T^{p,q}(V)$ and $(T^{p,q})'(V)$, and we can identify the two spaces along that isomorphism.

More precisely, the multifunctor $\mathcal{T}^{p,q}$ of 5.1 is naturally isomorphic to the multifunctor $(\mathcal{T}^{p,q})'$ given by

$$(\mathcal{T}^{p,q})'(V_1, \ldots, V_p, W_1, \ldots, W_q) \quad = \quad (V_1^* \otimes \ldots \otimes V_p^* \otimes W_1 \otimes \ldots \otimes W_q)^*$$

and this natural isomorphism induces a preferred isomorphism between $T^{p,q}(V)$ and $(T^{p,q})'(V)$.

Isomorphisms 5.2.3. The natural isomorphisms described in (5.2.2) yield isomorphisms from $T^{p,q}$ to various other vector spaces. For example (and this list is not exhaustive), we can construct (using 4.12) natural isomorphisms

$$\mathcal{T}^{1,1}(V,W) = V \otimes W^* \tag{5.2.3.1}$$

$$\approx (V^* \otimes W)^* \tag{5.2.3.2}$$

$$\approx \mathrm{Hom}(V^*, W^*) \tag{5.2.3.3}$$

$$\approx \mathrm{Hom}(W, V) \tag{5.2.3.4}$$

and these specialize to isomorphisms

$$
\begin{aligned}
T^{1,1}(V) &= V \otimes V^* && (5.2.3.1') \\
&\approx (V^* \otimes V)^* && (5.2.3.2') \\
&\approx \mathrm{Hom}(V^*, V^*) && (5.2.3.3') \\
&\approx \mathrm{Hom}(V, V) && (5.2.3.4')
\end{aligned}
$$

For another example, we have a series of natural isomorphisms

$$
\begin{array}{ccccccccccc}
\mathrm{Hom}(V, W^*) & \approx & (V \otimes W)^* & \approx & V^* \otimes W^* & \approx & W^* \otimes V^* & \approx & (W \otimes V)^* & \approx & \mathrm{Hom}(W, V^*) \\
 & & & & \| & & & & & & \\
 & & & & \mathcal{T}^{0,2}(V, W) & & & & & &
\end{array}
$$

Taking $W = V$, we get natural isomorphisms

$$
\begin{array}{ccccccccccc}
\mathrm{Hom}(V, V^*) & \approx & (V \otimes V)^* & \approx & V^* \otimes V^* & \approx & V^* \otimes V^* & \approx & (V \otimes V)^* & \approx & \mathrm{Hom}(V, V^*) \\
 & & & & \| & & & & & & \\
 & & & & T^{0,2}(V) & & & & & &
\end{array}
$$

where neither the center map nor the composition is the identity.

Exercise 5.2.3.5. Check that the center map takes $\sum f_i \otimes g_i$ to $\sum g_i \otimes f_i$, and the composition takes the map $f : V \to V^*$ to the map ϕ_f defined by $\phi_f(w)(v) = f(v)(w)$.

Exercise 5.2.3.6. List as many vector spaces as you can that can be called "naturally isomorphic" to $T^{1,2}(V)$.

Naturality and Abuse of Language 5.2.4. Are the isomorphisms (5.2.3.1′) – (5.2.3.4′) natural? Terms like $V \otimes V^*$ and $\mathrm{Hom}(V, V)$ are not even functorial in V, so the very question of naturality makes no sense. On the other hand, the isomorphisms are in no sense arbitrary; they are induced by the natural isomorphisms (5.2.3.1) – (5.2.3.4). That is, each (5.2.3.x) consists of isomorphisms $\phi_{V,W}$ for all V and W, and (5.2.3.x') is just the map $\phi_{V,V}$.

Thus there is a strong temptation to abuse language by calling the isomorphisms (5.2.3.x') "natural", and we will occasionally yield to that temptation.

Remark 5.2.5. For any basis of V, there is a dual basis given by (2.6.1), and from these bases you can construct a basis of $T^{p,q}(V)$ by repeated application of (2.13) together with the natural associativity of the tensor product (4.9). Any (p,q)-tensor (i.e. any element of $T^{p,q}(V)$) can be written in terms of these basis elements. If you choose a different basis for V, you'll get a different basis for $T^{p,q}(V)$, and a different expression for the same tensor. Many books emphasize the transformation rules for converting tensors from one basis to another. But in accordance with our philosophy that it's better to avoid choosing bases whenever possible, the transformation rules are of relatively minor interest.

Definition 5.2.6. For positive p and q we define a family of maps called *contractions* $T^{p,q}(V) \to T^{p-1,q-1}(V)$.

There is a contraction for each (i,j) with $0 \leq i \leq p$ and $0 \leq j \leq q$.

$T^{p,q}(V)$ is generated by terms of the form

$$v_1 \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_q^* \tag{5.2.6.1}$$

The (i,j) contraction maps the element (5.2.6.1) to

$$v_j^*(v_i)(v_1 \otimes \ldots v_{i-1} \otimes v_{i+1} \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots v_{j-1}^* \otimes v_{j+1} \otimes \ldots \otimes v_q^*)$$

Remark 5.2.6.2. The $(1,1)$ contraction

$$T^{1,1}(V) \to T^{0,0}(V) = \mathbf{R}$$

is just the trace map of (2.18).

6. Inner Products and Orthonormality

Throughout this section, V is a vector space and g is an element of $T^{0,2}V = V^* \otimes V^*$.

6A. Inner Products

Abuse of Language 6.1. From (5.2.3) we have a series of natural isomorphisms

$$\begin{array}{ccccccccccc} \mathrm{Hom}(V, V^*) & \approx & (V \otimes V)^* & \approx & V^* \otimes V^* & \approx & V^* \otimes V^* & \approx & (V \otimes V)^* & \approx & \mathrm{Hom}(V, V^*) \\ & & & & \| & & & & & & \\ & & & & T^{0,2}(V) & & & & & & \end{array}$$

(The map in the center is the map $\phi_{(V^*,V^*)}$ of (4.8).)

We will abuse notation by using the same symbol g to denote any of the pre-images of g and the a single symbol g^* to denote any of the images of g, thus:

$$\begin{array}{ccccccccccc} \mathrm{Hom}(V, V^*) & \approx & (V \otimes V)^* & \approx & V^* \otimes V^* & \approx & V^* \otimes V^* & \approx & (V \otimes V)^* & \approx & \mathrm{Hom}(V, V^*) \\ g & \mapsto & g & \mapsto & g & \mapsto & g^* & \mapsto & g^* & \mapsto & g^* \end{array}$$

In particular, g is identified with an element of $(V \otimes V)^* = \mathrm{Hom}(V \otimes V, \mathbf{R})$, and in view of (2.17), it can also be identified with a bilinear map $V \times V \to \mathbf{R}$. We will abuse language further by calling this map g as well.

Definition 6.2. We say that g is *symmetric* if

$$g(v, w) = g(w, v)$$

for all $v, w \in V$ (where g is identified with a bilinear map per (6.1)).

Exercise 6.2.1. Show that $g \in V^* \otimes V^*$ is symmetric if and only if $g = g^*$.

Definition 6.3. Using (6.1) to identify g with an element of $\mathrm{Hom}(V, V^*)$, we say that g is *nondegenerate* if the map $g : V \to V^*$ is an isomorphism.

Exercise 6.3.1. Show that g is nondegenerate if and only g^* is nondegenerate.

6.3.2. If we specify a nondegenerate g, we have an isomporphism $V \to V^*$; identifying V with V^* along this isomorphism, we automatically get identifications

$$T^{n,0}(V) \approx T^{n-1,1}(V) \approx \ldots \approx T^{0,n}(V)$$

As in (5.2.3), it makes no sense to ask whether these isomorphisms are natural because the $T^{p,q}$ are not even functorial. But also as in (5.2.3), we shall abuse language and treat these isomorphisms as if they "naturally identify" the various $T^{p,q}$ with $p+q=n$. It is important to realize, though, that these identifications depend on the choice of g.

Definition 6.4. g is called an *inner product* if it is symmetric and nondegenerate.

Warning 6.4.1. This definition is not entirely standard. Some authors require an "inner product" to satisfy additional axioms that we have not required here.

Remark and Notation 6.5. Note that $T^{0,2}$ is a contravariant bifunctor. (In general, $T^{p,q}$ is a functor when and only when $p=0$ or $q=0$.) Thus, given an inner product g on V and a linear transformation $f : U \to V$, we can consider the element $T^{0,2}(f)(g) \in T^{0,2}(U)$. We will abbreviate $T^{0,2}(f)(g)$ as $f^*(g)$.

In general, $f^*(g)$ need not be an inner product because it need not be nondegenerate. But we do have:

Proposition and Definitions 6.6. Let $i : U \to V$ be an injective linear transformation, and g an inner product on V. Then $i^*(g)$ is an inner product on U, called the *pullback* of **g** to U.

In case U is a subset of V and i is the inclusion map, we also call $i^*(g)$ the *restriction* of g to U and write it as $g|_U$.

Proposition 6.7. Let $U \subset V$ be a subspace and g an inner product on V. Identify g with a bilinear map on $V \times V$ (a la 6.1) and i^*g with a bilinear map on $U \times U$. Show that for any $u_1, u_2 \in U$ we have

$$i^*(g)(u_1, u_2) = g(u_1, u_2)$$

Definition 6.8. Let V be a vector space and g an inner product. For $v \in V$ we define the *norm* of v to be

$$\|v\| = |g(v,v)|^{1/2}$$

Note that the norm $\|v\|$ depends on g, though this dependence is suppressed in the notation.

Definitions 6.9. An inner product g is *negative definite* if $g(v,v) < 0$ for all $v \neq 0$.

An inner product g is *positive definite* if $g(v,v) > 0$ for all $v \neq 0$.

Definitions 6.10. Let V be a vector space, U a subspace, and g an inner product. We say that g is *negative definite on* U if the restriction of g to U is negative definite. There is always at least one such subspace, namely $\{0\}$. By (1.17.1) the dimension of U is never greater than the dimension of V, which is assumed to be finite by (1.7). Thus there is a largest natural number m such that there exists an m-dimensional subspace $U \subset V$ on which g is negative definite. We call m the *signature* of g.

We say that g is *positive definite* on U if the restriction of g to U is positive definite.

Proposition 6.11. Suppose $V = N \oplus P$. Treat N and P as subspaces of V via (1.15.1). Suppose that g is negative definite on N and positive definite on P. Then the signature of g is equal to the dimension of N.

Proof. Let X be a subspace on which g is negative definite. Note that $X \cap P$ must equal $\{0\}$. Claim: X cannot contain two distinct elements of the form (n,p) and (n,p'). Proof of claim: Otherwise $(0, p-p') \in X \cap P$, contradiction. It follows that the map $X \to N$ defined by $(n,p) \mapsto n$ is injective, so X is isomorphic to a subspace of N, hence has dimension at most that of N by (1.17).

6B. Orthonormality

Definition 6.12. Let g be an inner product on V. A basis $\{v_1, \ldots, v_n\}$ is called *orthonormal* with respect to g if for some m we have

$$g(v_i, v_j) = \begin{cases} -1 & \text{if } i = j \leq m \\ 1 & \text{if } i = j > m \\ 0 & \text{if } i \neq j \end{cases}$$

Proposition 6.12.1. If $\{v_1, \ldots, v_n\}$ is orthonormal, then the integer m in (6.12) must be equal to the signature of g.

Proof. Let N be the subspace generated by $\{v_1, \ldots, v_m\}$, let P be the subspace generated by $\{v_{m+1}, \ldots, v_n\}$, and apply (6.10).

Proposition 6.12.2 For any inner product g, there exists a basis that is orthonormal with respect to g.

Proof. Start with an arbitrary basis $\{v_i\}$.

Claim 6.12.2.1. We can assume that $g(v_i, v_i) \neq 0$ for every i.

Proof. Given i, nondegeneracy implies that there exists a j with $\mathbf{g}(v_i, v_j) \neq 0$. Let ϵ be any real number and set $v_i' = v_i + \epsilon v_j$. Replace v_i with v_i' and check that $\{v_i, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_n\}$ is still a basis. Moreover,

$$g(v_i', v_i') = g(v_i, v_i) + 2\epsilon g(v_i, v_j) + \epsilon^2 g(v_j, v_j)$$

which is nonzero for all but at most two choices of ϵ.

Proof of 6.12.2, completed. Let r be the largest integer such that $g(v_i, v_j) = 0$ for all $i \neq j < r$.

If $r \leq n$, replace v_r by

$$v_r' = v_r - \sum_{i=1}^{r-1} \frac{g(v_i, v_r)}{g(v_i, v_i)} v_i$$

(Note that the quotient makes sense because of (6.12.2.1).) Check that this replacement does not change the fact that the v_i form a basis, and it increases r by 1. Thus by induction we can assume that $g(v_i, v_j) = 0$ for all $i \neq j$.

Now multiply each v_i by an appropriate constant to make all the squared lengths equal to ± 1, and rearrange so that all of those with negative squared lengths come first.

Notation 6.13. In working with examples, it is sometimes useful to choose a basis

$v_1, \ldots, v_n$ and to introduce the array

$$\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix} \tag{6.13.1}$$

where $g_{ij} = g(v_i \otimes v_j)$ (here g is identified with an element of $(V \otimes V)^*$ per (6.1)).

The array (6.13.1) is called the *matrix of g with respect to the basis* $\{v_1, \ldots, v_n\}$.

Exercise 6.13.2. Show that g is symmetric if and only if $g_{ij} = g_{ji}$ for all i and j.

Remark 6.13.3. It is an immediate consequence of the definitions that the basis $\{v_1, \ldots, v_n\}$ is orthonormal with respect to g if and only if the matrix of g with respect to $\{v_1, \ldots, v_n\}$ is

$$\begin{pmatrix} -1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

It is an immediate consequence of (6.12.1) that the number of minus ones in this representation is equal to the signature of g.

Defintion 6.14. Let $U \subset V$ be a subspace. We define

$$U^\perp = \{v \in V | g(v, u) = 0 \text{ for all } u \in U\}$$

The notation suppresses the fact that $U^\perp$ depends on g.

Exercise 6.14.1. Show that $U^\perp$ is a subspace of V.

Proposition 6.15. Let V be a vector space with inner product g. Then V contains subspaces N and P such that

i) g is negative definite on N and positive definite on P.

ii) $N^\perp = P$ and $P^\perp = N$

iii) $N \cap P = 0$

iv) $V \approx N \oplus P$

Proof. Choose an orthonormal basis $\{v_1, \dots, v_n\}$. Let σ be the signature of g, so that $g(v_i, v_i) = -1$ for $i \leq \sigma$ and $g(v_i, v_i) = 1$ for $i > \sigma$. Let N be the set of all linear combinations of $v_1, \dots, v_\sigma$ and let P be the set of all linear combinations of $v_{\sigma+1}, \dots, v_n$.

For $v = \sum_{i=1}^{\sigma} \alpha_i v_i \in N$, we have $g(v, v) = -\sum_{i=1}^{\sigma} \alpha_i^2$, so g is negative definite on N. Similarly, g is positive definite on P.

(iii) follows from uniqueness of the representation of elements of v as linear combinations of the v_i.

For (iv), map $N \oplus P \to V$ by $(v, w) \mapsto v + w$ and check that this is an isomorphism.

6C. Some Inequalities

Here we record some inequalities that will be needed in Chapter Four.

Proposition 6.16 (The Schwarz Inequality). Suppose g is either negative definite or positive definite. Then for any vectors $v, w \in V$, we have

$$g(v, v)g(w, w) \geq g(v, w)^2$$

Proof. If $v = 0$, both results are obvious. Now assume $v \neq 0$. Let

$$x = g(v, v)w - g(v, w)v$$

and compute that

$$g(x, x) = g(v, v)^2 g(w, w) - g(v, v)g(v, w)^2$$

The hypothesis of negative or positive definiteness implies that

$$\begin{aligned} 0 &\leq \frac{g(x, x)}{g(v, v)} \\ &= \frac{g(v, v)^2 g(w, w) - g(v, v)g(v, w)^2}{g(v, v)} \\ &= g(v, v)g(w, w) - g(v, w)^2 \end{aligned}$$

(Make sure you see where this calculation uses the symmetry of g!)

Definition 6.17. An element $v \in V$ is *timelike* if $g(v, v) < 0$.

Proposition 6.18. Suppose the signature of g is 1. Let $n \in V$ be the first element of an orthonormal basis, so $g(n, n) = -1$). Then for any timelike $u, v \in V$ we have

$$g(u, v)g(u, n)g(v, n) > 0$$

Proof. First, let N and P be as in (6.15). Note that N is one- dimensional (because the signature of g is 1) and so that by (1.17.2), the singleton set $\{n\}$ is a basis for N. In other words, every element of N is a scalar multiple of n.

For $v \in V$, set

$$x_v = v + g(v, n)n$$

Then an easy computation gives $g(x_v, \alpha n) = 0$ for any real number α. It follows from (6.15ii) that $x_v \in P$ so that

$$\begin{aligned} 0 &< g(x_v, x_v) && \text{(positive definiteness)} \\ &= g(v, v) + g(v, n)^2 && \text{(a computation)} \\ &< g(v, n)^2 && \text{(because } v \text{ is timelike)} \end{aligned}$$

and similarly

$$0 < g(x_u, x_u) < g(u, n)^2$$

This gives

$$\begin{aligned} g(u, n)^2 g(v, n)^2 &> g(x_u, x_u)g(x_v, x_v) \\ & geg(x_u, x_v)^2 && \text{(by (6.16))} \\ &= \big(g(u, v) + g(u, n)g(v, n)\big)^2 && \text{(a computation)} \\ &= g(u, n)^2 g(v, n)^2 + 2g(u, v)g(u, n)g(v, n) + g(u, v)^2 \end{aligned}$$

so that

$$0 > 2g(u, v)g(u, n)g(v, n) + g(u, v)^2 > 2g(u, v)g(u, n)g(v, n)$$

as needed.

Corollary 6.19. Suppose the signature of g is 1. Let $u, v, w \in V$ be timelike. Then

$$\frac{g(u,v)g(v,w)}{g(u,w)} > 0$$

Proof.

$$\frac{g(u,v)g(v,w)}{g(u,w)} = \frac{\big(g(u,v)g(u,n)g(v,n)\big)\big(g(v,w)g(v,n)g(w,n)\big)}{\big(g(u,w)g(u,n)g(w,n)\big)} \times \frac{1}{g(v,n)^2}$$

Definition 6.20. Two timelike vectors $u, v \in V$ are said to have the same *time orientation* if $g(u,v) < 0$.

Proposition 6.21. If the signature of g is 1, then time orientation is an equivalence relation on the set of all timelike vectors.

Proof. Reflexivity follows from the definition of timelike. Symmetry follows from the symmetry of g. Transitivity follows from (6.19).

Proposition 6.22.

i) Let v and v' be timelike vectors. Then v is equivalent to exactly one of v' or $-v'$.

ii) Time orientation divides the set of all timelike vectors into exactly two equivalence classes.

Proof. Statement (ii) follows from Statement (i), which in turn follows from the equality

$$g(v, -v') = -g(v, v')$$

Definition 6.23. A vector $v \in V$ is *lightlike* if $v \neq 0$ and $g(v,v) = 0$.

Remarks 6.24. We would like to extend the definition of time orientation so that it applies to lightlike vectors as well as timelike vectors. Unfortunately, a naive extension of the definition (v and w have the same time orientation if $g(v,w) < 0$) doesn't work because then time orientation would not be a reflexive relation for lightlike vectors. Relaxing the definition to allow $g(v, w) \leq 0$ doesn't work either because it leads to violations of transitivity. So we make the following definition:

Definition 6.25. Let v be timelike and let p be lightlike. Then p and v have the *same time orientation* if $g(p, v) < 0$.

Let p and q be lightlike. Then p and q have the *same time orientation* if there exists a timelike vector v that has the same time orientation as both p and q.

Exercise 6.26. Suppose that g has signature 1 and let N be as in (6.15). Let $\{n\}$ be a basis for N as in the proof of (6.18). Let v and w be vectors, each of which is either timelike or lightlike. Show that v and w have the same time orientation if and only if

$$g(v, n)g(w, n) > 0$$

Proposition 6.27. Time orientation is an equivalence relation on the set of all timelike and lightlike vectors. There are exactly two equivalence classes for this relation.